AD 420013

DEFENSE DOCUMENTATION CENTER

FOR

SCIENTIFIC AND TECHNICAL INFORMATION

CAMERON STATION, ALEXANDRIA, VIRGINIA

AIR FORCE INSTITUTE OF TECHNOLOGY

AIR UNIVERSITY
UNITED STATES AIR FORCE

THE VERNIER DEVICE AND
RESIDUE NUMBER SYSTEMS

THESIS

GE/EE/63-7

Richard E. Evans
Capt. USAF

SCHOOL OF ENGINEERING

WRIGHT-PATTERSON AIR FORCE BASE, OHIO

AF-WP-O-OCT 62 3,500

THE VERNIER DEVICE
AND
RESIDUE NUMBER SYSTEMS

THESIS

Presented to the Faculty of the School of Engineering of
the Air Force Institute of Technology
Air University
in Partial Fulfillment of the
Requirements for the
Master of Science Degree
in Electrical Engineering

By

Richard Earl Evans, B.S.E.E.
Capt USAF

Graduate Electronics

August 1963

Preface

I first became aware of residue numbers and their potential while attending the thesis presentations of class GE 62. The work done by Capt. Arthur J. Altenburg of that class aroused in me considerable interest in this unique number system. Thus, when the Computer Section, Bionics and Computer Branch, Electronic Technology Laboratory, Aeronautical Systems Division, suggested to my class the subject of analog-to-residue number conversion as a possible thesis topic I was immediately enthusiastic.

As originally conceived, my thesis project was simply one of adaption. I had felt that a suitable result would be obtained if standard analog-to-digital conversion techniques were modified so as to be applicable to residue numbers. With this goal, I began to study present day conversion methods. During a discussion of some results of my study, Mr. Dewey E. Brewer, the ASD sponsor for my thesis, suggested the possibility of using vernier devices. To determine the feasibility of this idea, I constructed a simple model. This model showed that a definite relationship exists between residue numbers and the coding system used on a vernier device. Because of this discovery, I abandoned my original idea and changed my thesis project to an investigation of the relationship between vernier devices and residue numbers. This paper gives the results of that investigation.

I want to emphasize that practical application is not the

primary intention of this paper. The main excuse for its existence (besides the obvious one) is the possibility that it might provide some slight additional insight into the properties of residue numbers. However, I could not help but visualize certain applications. Some of the more reasonable speculations are included in the recommendations of Chapter VI.

The work presented in this paper is to the best of my knowledge my own. As a result, any errors that appear must of necessity be brought to my doorstep.

At this time I wish to add a few personal notes. First, I wish to acknowledge my indebtedness to the United States Air Force for affording me the opportunity of continuing my education, and to the Air Force Institute of Technology for providing me with the means of taking advantage of that opportunity. Next, I want to express my appreciation to Mr. Brewer, who so kindly sponsored my work; to Capt. James E. McCormick, my Faculty Thesis Advisor, whose expressed confidence in my work gave me the extra incentive needed to overcome the bad moments that occur in a work of this nature; and, to Capt. Frank M. Brown, who as a teacher and as a friend provided invaluable guidance. To these people I offer my most heartfelt thanks.

Finally, I must express my indebtedness to my wife. Her constant encouragement and cooperation made this work possible. Not least among the many things she did for me was the typing of the manuscript.

Richard E. Evans

Contents

Contents

List of Figures

List of Figures

Abstract

Properties of vernier devices and residue number systems are investigated. The investigation is performed in two parts. First, theory for conventional verniers is developed. This theory includes: necessary conditions for constructing a vernier device, basic equation describing vernier operation, vernier scale coding sequence, and characteristics of that sequence. Second, properly designed devices, called residue verniers, are shown to provide decimal (analog) to residue number conversion. The developed device theory includes: necessary conditions to prevent ambiguity in the number system, sufficient conditions to prevent ambiguity of scale readings, physical parameters of the device, and coding sequences for the various scales.

THE VERNIER DEVICE

AND

RESIDUE NUMBER SYSTEMS

I. Introduction

It was in 1955 that Americans first became aware of the work being done by two Russians, M. Valach and A. Svoboda, on residue number systems. Their work, which shows the possibiities of the application of residue numbers and the associated arithmetic to computing machinery, has stimulated considerable activity in this country. The culmination of this activity will be the completion at the end of this year of an experimental computer based on the residue number concept.

The property of residue numbers that has generated all this interest has to do with the carry operation in arithmetic. In present day computer coding systems, the necessity of handling the carry results in limiting the speed of operation. Residue number systems offer a solution to this problem because the notion of carry is not involved in the algorithms for addition and multiplication. This makes possible the construction of economical arithmetic circuits capable of very high speed operation. However, before residue number systems can be successfully applied to general purpose machines, efficient means must be derived to overcome several difficulties.

These difficulties include: the determination of relative magnitude, the determination of algebraic sign, round-off, and division (Ref 1: Chap. III, p. 3). As a result of these and other problems, there is still a definite need for further theoretical development of the properties of residue number systems. This paper is an attempt to contribute in a small way to this need.

The purpose of this paper is to investigate the properties of residue number systems and of vernier devices. The intention of this investigation is to show that these seemingly unrelated subjects are, in fact, very closely related. To make this relationship manifest, mathematical expressions are developed which show that given certain defined parameters, a vernier device may be constructed so that its scales provide readings in a given residue number system - the readings being the residue number representation of the decimal distance measured by the device.

To accomplish the above purpose, this paper is basically separated into three parts. In the first part (Chapter II), a theory of conventional vernier devices is developed. This theory is not specifically applicable to the use of vernier devices with residue number systems. However, its development is necessary in order to provide a starting place for the development of a theory that does apply.

The second part of this paper (Chapters III, IV, and V) is concerned with the development of the vernier device, residue number relationships. Chapter III presents a brief and limited introduction to the concept of residue numbers. The introduction is intended only to prepare the reader for the subsequent development in Chapters IV and V. Chapter IV is the heart of this paper. It presents the theoretical development for the use of residue numbers with a basic two-scale vernier device. The results of this development are the sufficient conditions for the design and the requirements for construction of the basic two-scale device. The device so designed and constructed is referred to as a "residue vernier". Subsequently, (Chapter V), the theory of the two-scale device is extended and generalized so that it applies to the more complex multi-scale devices. This leads to the generalized sufficient conditions and requirements for the design and construction of residue verniers.

The third and final part of this paper (Chapter VI and VII) presents a summary of the pertinent results of the investigation and gives recommendations for further work in this area.

II. Conventional Verniers

The vernier device was first described by its inventor, Piere Vernier, in 1631. As conceived by him, the device provides a method for increasing the accuracy with which scales may be read. This chapter will discuss Vernier's device from the standpoint of its conventional use. A later chapter will introduce a new and different mode of application, the use of vernier devices with residue number systems.

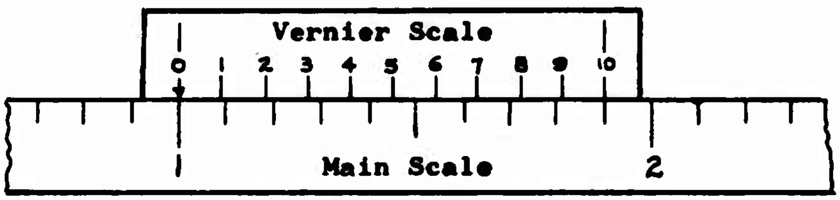
The discussion in this chapter will be in two sections. The first section will briefly describe conventional vernier devices and will give a few simple illustrations of their use. This section is primarily intended to be a review that will refamiliarize the reader with the operation of conventional verniers. Though such devices are not the direct concern of this paper, their review should enable the reader to better follow the subsequent theoretical development. The second section of this chapter will consist of the author's development of the theory of operation of conventional vernier devices. This development will be general in nature, and will not be specifically applicable to the use of verniers with residue number systems. However, the development of this theory is necessary in order to provide the insight required to associate vernier devices with residue numbers.

Conventional Vernier Devices

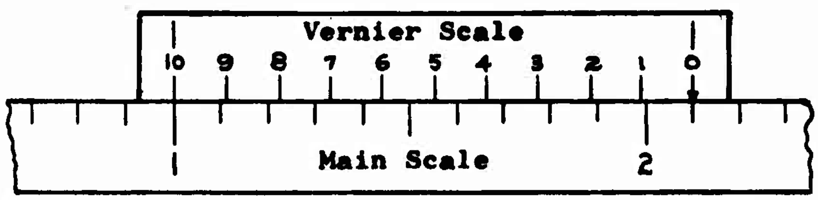
Conventional vernier devices consist of an auxiliary (vernier) scale sliding in contact with the scale to be read (main scale). The vernier scale is divided into N_v intervals which occupy the same space as $N_v \pm 1$ intervals on the main scale. Each interval of the vernier scale will then occupy $1 \pm \frac{1}{N_v}$ intervals of the main scale. When the relation $N_v + 1$ is used, it is necessary for the vernier and main scales to read in opposite directions. This type of device is known as a retrograde or reverse vernier. When the relation $N_v - 1$ is used both scales read in the same direction, and the device is known as a direct vernier.

Linear Verniers. Scales used to provide linear measurements are generally designed to give readings in decimal form. To obtain decimal readings the vernier scale is divided into ten intervals. The combined length of these ten intervals is made equal to the length of either nine or eleven intervals on the main scale depending on the type of vernier desired. Figure 1 illustrates both types of linear verniers.

Angular Verniers. Scales used to measure angles may be designed to read decimal parts of degrees by using the same interval relationships as described for linear scales. Angular scales designed to read in minutes or seconds of arc, however,



(a) Direct Vernier



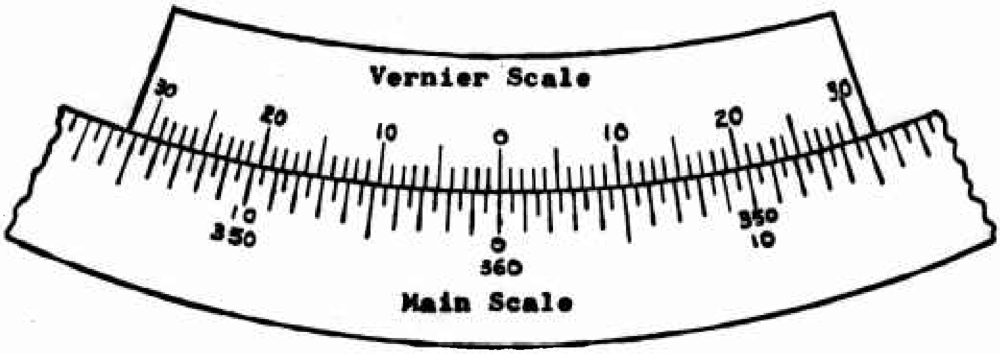
(b) Retrograde Vernier

Fig. 1

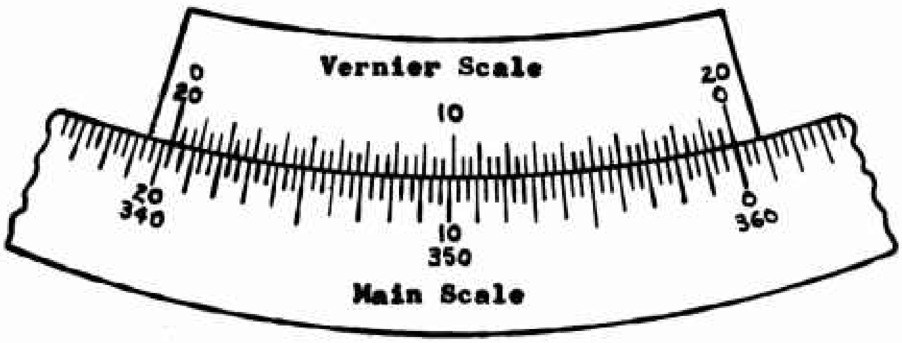
Linear Verniers

require a different relationship. Also, since it may be desirable to measure angles in either direction, the verniers are usually double (i.e., a single vernier is placed on each side of the fiducial, one of which is to be used in reading angles to the right, and the other in reading angles to the left). Figure 2(a) illustrates a typical direct angular vernier used to read to one minute of arc. This device is constructed by dividing the main scale into degrees and half-degrees. The vernier scale is then made to provide one minute readings by taking a length equal to 29 of the half-degree intervals and subdividing it into 30 equal parts.

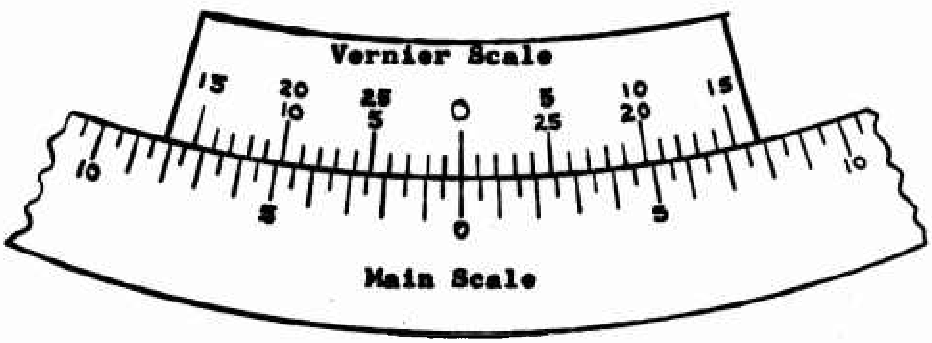
For some angular verniers the length of the scale makes it impracticable to use a double vernier. If it is still desirable to read angles in either direction, a single vernier with two rows of numbers may be used as shown in Fig. 2(b). It is evident that if angles are to be read clockwise the right fiducial should be used; whereas, angles to be read counterclockwise require the use of the left fiducial. To construct this device the main scale is divided into 20' spaces. The vernier scale is then made to provide 20" readings by taking a length equal to 59 of the 20' spaces and subdividing it into 60 equal parts.



(a) Double Vernier Reading To 1'



(b) Single Vernier Reading to 20"



(c) Folded Vernier Reading to 1'

Fig. 2

Direct Angular Verniers

The inconvenience of two fiducials may be overcome by using the middle line as the fiducial and "folding" the scale as shown in Fig. 2(c). This "folded" vernier is read like an ordinary vernier except that if coincidence is not reached by passing along the vernier in the direction in which the main scale is numbered, it is necessary to go to the other end of the vernier and continue in the same direction, toward the center, until coincidence is found. The design relation for this device is the same as that used for the vernier illustrated in Fig. 2(a).

Theory of Conventional Verniers

Definitions and Conditions. As previously mentioned, the following development of vernier device theory is general in nature, and is not directed toward the ultimate use of vernier devices with residue number systems. However, it does lay down the necessary groundwork for the development of that use.

A conventional vernier device is normally thought of as consisting of two scales, a main scale and a vernier scale. However, for the following development it is necessary to consider a vernier device as made-up of three scales: a primary scale consisting of the primary divisions of the main scale, a secondary scale consisting of the subdivisions of the primary scale, and a vernier scale. Also, for this development it will

be convenient to use the length of a single vernier scale as a reference. For this purpose, this length is defined as a "vernier scale cycle" (e.g., the double vernier scale in Fig. 2(a) is a two-cycle scale).

With the above provisions in mind, the following symbols are defined:

N_{sp} = number of intervals on the secondary scale per primary scale division

N_{sv} = number of intervals on the secondary scale per vernier scale cycle

N_v = number of intervals on the vernier scale per vernier scale cycle

L_p = length of one primary scale division

L_s = length of one secondary scale interval

L_v = length of one vernier scale interval

From practical considerations it may be seen that N_{sp}, N_{sv}, and N_v must be integers. Also, in order to prevent any ambiguity of coincidence between the secondary and vernier scales, it is necessary that N_{sv} and N_v be relatively prime.

<u>Distance Equation</u>. When there is coincidence between the secondary and vernier scales, as shown in Fig. 3, then a distance equation may be written such that

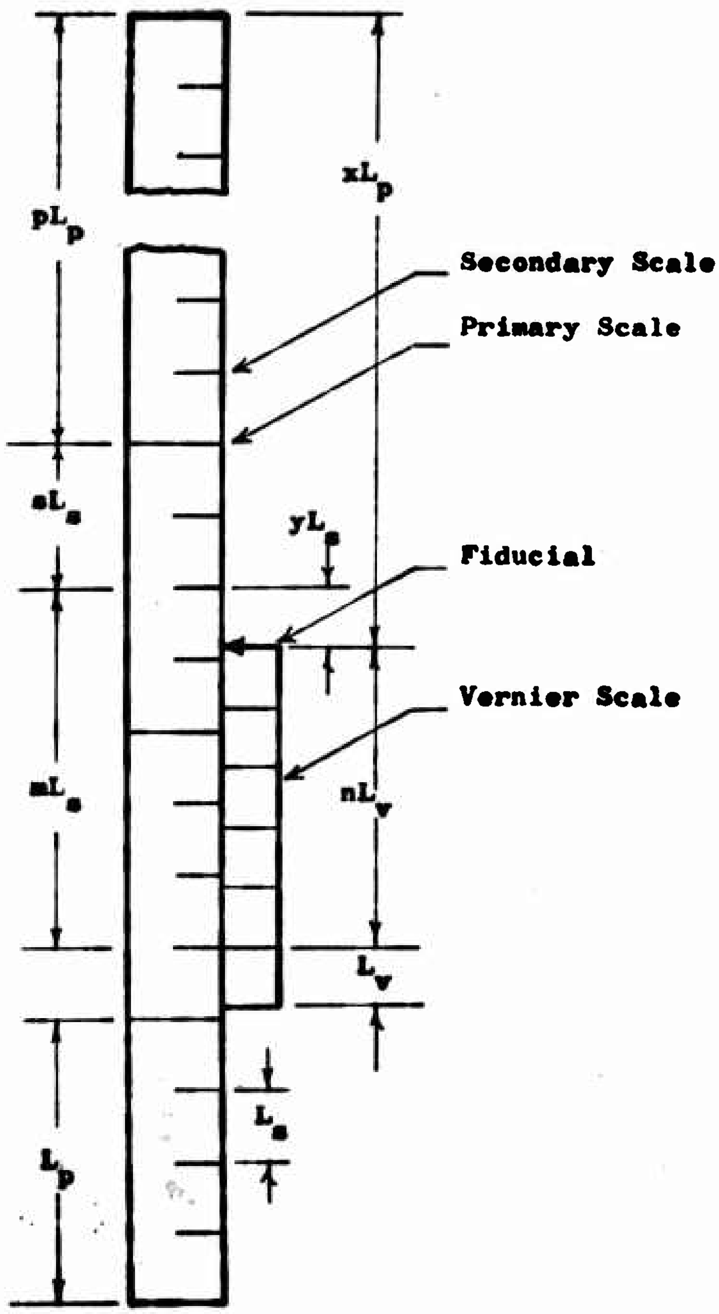


Fig. 3
Vernier Device Parameters
For Distance Equation

$$xL_p = pL_p + sL_s + mL_s - nL_v \tag{1}$$

where xL_p = distance between primary scale zero and fiducial

pL_p = largest integral multiple of L_p between primary scale zero and fiducial

sL_s = largest integral miltiple of L_s between last primary division and fiducial

mL_s = distance between last secondary division and coincidence of scales

nL_v = distance between coincidence of scales and ficucial

It should be noted that the set p, s, m, and n is a unique function of x, and with coincidence it is a set of integers.

From the definitions in the preceding subsection it may be seen that

$$L_s = \frac{L_p}{N_{sp}} \tag{2}$$

and

$$L_v = \frac{N_{sv}L_s}{N_v} = \frac{N_{sv}L_p}{N_{sp}N_v} \tag{3}$$

The substitution of Eqs (2) and (3) into Eq (1) gives

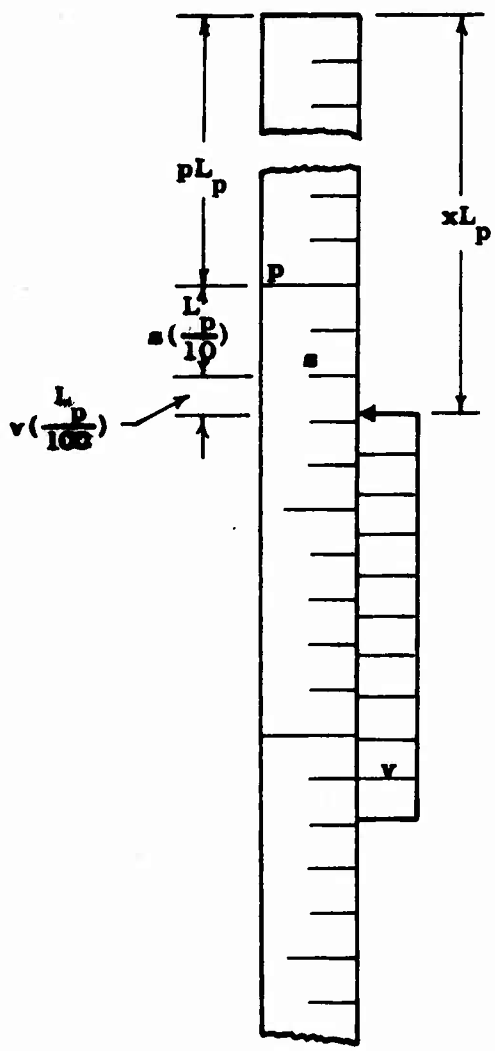


Fig. 4

Conventional Linear Vernier
($N_{sp} = 10$, $N_v = 10$)

$$xL_p = pL_p + s\left(\frac{L_p}{N_{sp}}\right) + (mN_v - nN_{sv})\frac{L_p}{N_{sp}N_v} \qquad (4)$$

The meaning of Eq (4) may be better understood by use of an example. From Fig. 4 it may be seen that if the length L_p is one inch, then

$$\begin{aligned} x\ (\text{inches}) &= p\ (\text{inches}) + s\ (\text{tenths of an inch}) \\ &\quad + v\ (\text{hundredths of an inch}) \end{aligned} \qquad (5)$$

where p, s, and v are the respective readings of the primary, secondary, and vernier scales. For the conventional linear vernier illustrated, $N_{sp} = 10$ and $N_v = 10$. Substitution of these values and $L_p = 1$ inch into Eq (4) gives

$$\begin{aligned} x\ (\text{inches}) &= p\ (\text{inches}) + s\ (\text{tenths of an inch}) \\ &\quad + (mN_v - nN_{sv})\ (\text{hundredths of an inch}) \end{aligned} \qquad (6)$$

Comparison of Eqs (5) and (6) shows that v, the reading of the vernier scale at coincidence, must be given by

$$v = mN_v - nN_{sv} \qquad (7)$$

The significance of this important relationship will be examined in detail in the next subsection. For now it is sufficient to realize that the use of Eq (7) allows Eq (4) to be written as

$$x = p + \frac{s}{N_{sp}} + \frac{v}{N_{sp}N_v} \tag{8a}$$

or

$$x = \frac{pN_{sp}N_v + sN_v + v}{N_{sp}N_v} \tag{8b}$$

where p, s, and v are the scale readings of the device.

Equation (8) is the basic equation behind the operation of vernier devices. It shows why it is possible, when using such a device for measurement, to take the readings directly from the various scales to obtain a numerical evaluation of the length (angle) being measured. Perhaps a better understanding of Eq (8) may be obtained by returning to the example illustrated in Fig. 4. For this device Eq (8) becomes

$$x = p + \frac{s}{10} + \frac{v}{10^2} = \frac{p(10^2) + s(10) + v}{10^2} \tag{9}$$

and takes on the familiar form of the base 10 (decimal) number system.

A relation indicated by Eq (8) is worthwhile noting, although any development of it is beyond the scope of this paper. It may be seen from Eq (8) that by appropriate choice of N_{sp} and N_v the value of x may be obtained in any desired fixed or mixed radix number system. From the viewpoint of this paper, however, the relation given by Eq (7) has greater consequence.

Vernier Scale Coding. To examine the meaning of Eq (7) it is first necessary to write another distance equation. From Fig. 3 it may be seen that

$$yL_s = mL_s - nL_v \tag{10}$$

where yL_s is defined as the distance from the last secondary division to the fiducial. Also, it may be seen that

$$0 \leq y \leq 1 \tag{11}$$

Substitution of Eqs (2) and (3) into Eq (10) gives

$$y = \frac{mN_v - nN_{sv}}{N_v} \tag{12}$$

The combination of Eqs (11) and (12) results in

$$0 \leq mN_v - nN_{sv} \leq N_v \tag{13}$$

It will be seen later that the difference $N_{sv} - N_v$ is a convenient parameter for verniers. With this in mind, the expression $mN_v - nN_{sv}$ may be rewritten as

$$mN_v - nN_{sv} = qN_v - nN \tag{14}$$

where

$$N = N_{sv} - N_v \tag{15}$$

$$q = m - n \tag{16}$$

Thus,

$$0 \leq qN_v - nN \leq N_v \tag{17}$$

An examination of the factors that form the expression $qN_v - nN$ gives the following: the factors N_v and N are constants when a specific vernier device is under consideration, the factor n is the number of intervals a given line on the vernier scale is from the fiducial, and the factor q is an integer. When N_v and N are given, the inequality of Eq (17) shows that the value of q is a function of n. Since q is a function of n, it follows from Eqs (7) and (14) that v is also a function of n. Thus, v, the numerical value given a particular line on the vernier scale, is directly related to n, the

number of intervals a given line on the vernier scale is from the fiducial. Since v will take on a sequence of values Eqs (7) and (14) may be combined and rewritten as

$$\{v_n\} = \{q(n)N_v - nN\} \quad (18)$$

where $n = 0,1,2, \ldots, N_v$ and $0 \leq v_n \leq N_v$

Because this sequence determines the numerical coding of the vernier scale, $\{v_n\}$ will be referred to as the coding sequence.

The direct and retrograde vernier scales previously discussed are special cases of Eq (18) where $\{v_n\}$ is a monotone sequence. The direct scale requires a sequence that is monotone increasing; the retrograde scale requires one that is monotone decreasing. In order for $\{v_n\}$ to be monotone, it is necessary that

$$v_{k+1} = v_k \pm 1 \quad (19)$$

The restriction of Eq (19) applied to Eq (18) gives

$$v_{k+1} - v_k = \left[q(k+1) - q(k)\right] N_v - \left[(k+1) - k\right] N \quad (20)$$

If $$r \triangleq q(k+1) - q(k) \quad (21)$$

then Eq (18) becomes

$$N = rN_v \mp 1 \quad (22)$$

Since q(n) is an integer it follows from Eq (21) that r is an integer. Also, Eq (21) indicates that r may be a function of n. However, Eq (22) presented in a little different form

$$r = \frac{N \mp 1}{N_v} \tag{23}$$

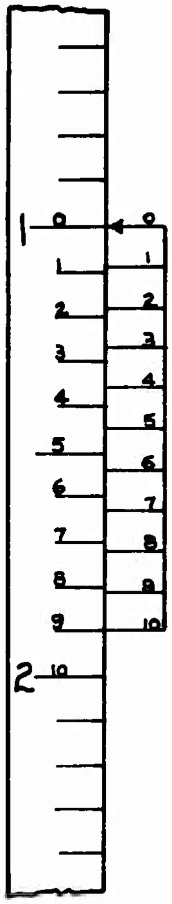
shows that r must be a constant for a given monotone sequential vernier. It will be proven in the following subsection that r may have the values

$$r = -1, 0, 1, 2, 3, \ldots \tag{24}$$

Thus, Eq (22) states that a vernier device with parameters such that N is equal to an integral multiple of N_v minus (plus) one will have a monotone sequence as a code. The minus sign in Eq (22) results in $\{v_n\}$ being a monotone increasing sequence (direct vernier), while the plus sign gives a monotone decreasing sequence (retrograde vernier). Figures 5 and 6 illustrate how the conventional vernier scales result from the above relations when $N = \mp 1$, $(r = 0)$.

When Eq (22) is not satisfied $\{v_n\}$ will be non-monotone. An example of a vernier scale coded by a non-monotone sequence is illustrated in Fig. 7. Though the parameter r is defined

$N_{sp} = 10$	$N_{sv} = 9$		
$N_v = 10$	$N = -1$		
n	$N_v q(n) - Nn$	v_n	$q(n)$
0	$10q(0)+0$	0	0
1	$10q(1)+1$	1	0
2	$10q(2)+2$	2	0
3	$10q(3)+3$	3	0
4	$10q(4)+4$	4	0
5	$10q(5)+5$	5	0
6	$10q(6)+6$	6	0
7	$10q(7)+7$	7	0
8	$10q(8)+8$	8	0
9	$10q(9)+9$	9	0
10	$10q(10)+10$	10	0

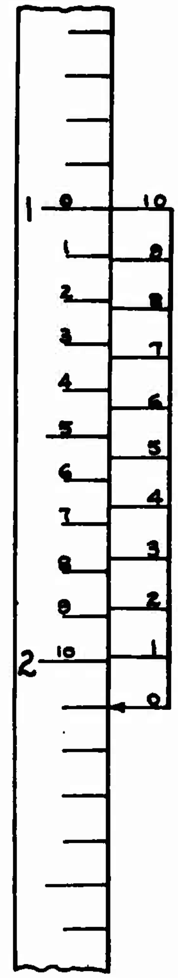


$$x = p + \frac{s}{10} + \frac{v}{10^2} = \frac{p(10^2)+s(10)+v}{10^2}$$

Fig. 5

Direct Vernier

$N_{sp} = 10$		$N_{sv} = 11$	
$N_v = 10$		$N = 1$	
n	$N_vq(n)-Nn$	v_n	q(n)
0	10q(0)-0	10	1
1	10q(1)-1	9	1
2	10q(2)-2	8	1
3	10q(3)-3	7	1
4	10q(4)-4	6	1
5	10q(5)-5	5	1
6	10q(6)-6	4	1
7	10q(7)-7	3	1
8	10q(8)-8	2	1
9	10q(9)-9	1	1
10	10q(10)-10	0	1

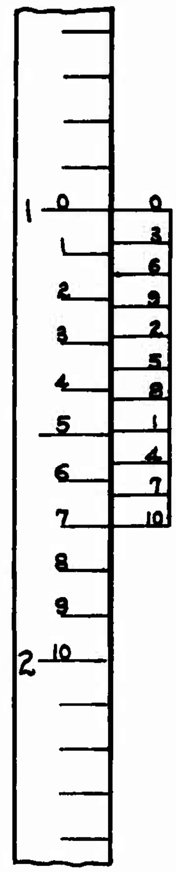


$$x = p + \frac{s}{10} + \frac{v}{10^2} = \frac{p(10^2)+s(10)+v}{10^2}$$

Fig. 6

Retrograde Vernier

$N_{sp} = 10$		$N_{sv} = 7$	
$N_v = 10$		$N = -3$	
n	$N_v q(n) - Nn$	v_n	$q(n)$
0	10q(0)+0	0	0
1	10q(1)+3	3	0
2	10q(2)+6	6	0
3	10q(3)+9	9	0
4	10q(4)+12	2	-1
5	10q(5)+15	5	-1
6	10q(6)+18	8	-1
7	10q(7)+21	1	-2
8	10q(8)+24	4	-2
9	10q(9)+27	7	-2
10	10q(10)+30	10	-2



$$x = p + \frac{s}{10} + \frac{v}{10^2} = \frac{p(10^2)+s(10)+v}{10^2}$$

Fig. 7

Folded Vernier

for monotone sequential verniers only, it may be noted from Fig. 7 that $q(k+1) - q(k)$ is not a constant for the non-monotone sequential vernier. Because of the lack of a better name - due, no doubt, to a lack of imagination - the author has chosen to refer to verniers coded by non-monotone sequences as "folded verniers". The reader is cautioned not to confuse the folded (non-monotone) vernier with the folded angular vernier discussed earlier. Because of the difficulty with interpolating non-coincident readings, folded verniers are of little practical value for use as conventional verniers designed to measure length (angle). However, folded verniers are important when vernier devices are used with residue numbers. As a result, they will be examined in more detail in the following subsection.

<u>Vernier Families and the Folded Vernier</u>. From Eq (11) it can be seen that the "least count" of a vernier (the smallest division that can be read directly from a vernier) is $\frac{L_p}{N_{sp} N_v}$.
Thus, for given values of N_{sp} and N_v, a family of vernier devices having the same least count may be constructed by assigning different values to N_{sv}. A given vernier family will consist of two subgroups: the monotone sequential verniers and the non-monotone sequential verniers. Figures 5 and 6 are examples of the former, while Fig. 7 is an example of the latter.

Subgroups in any family share common characteristics and interconnecting relationships. The vernier families are no exception. The subsequent development will show that a given non-monotone sequential vernier is directly related to a monotone sequential vernier of the same family. However, before beginning this development, it is first necessary to examine r of Eq (22) and prove Eq (24).

From the inequality of Eq (17) it may be seen that

$$\frac{Nn}{N_v} \leq q(n) \leq 1 + \frac{Nn}{N_v} \tag{25}$$

If n takes on the values k and k+1, then

$$-\left(1 + \frac{Nk}{N_v}\right) \leq -q(k) \leq -\frac{Nk}{N_v} \tag{26}$$

$$\frac{N(k+1)}{N_v} \leq q(k+1) \leq 1 + \frac{N(k+1)}{N_v} \tag{27}$$

The addition of Eqs (26) and (27) and the substitution of Eq (21) into the resulting sum gives

$$\frac{N}{N_v} - 1 \leq r \leq \frac{N}{N_v} + 1 \tag{28}$$

The substitution of the definition of N, Eq (15), allows Eq (28) to be rewritten as

$$\frac{N_{sv}}{N_v} - 2 \leq r \leq \frac{N_{sv}}{N_v} \tag{29}$$

In a given vernier family N_v is a constant and N_{sv} may take on any positive integer. As a result, for a given family, Eq (29) shows that r may take on the values as given in Eq (24)

$$r = -1, 0, 1, 2, 3, \ldots \tag{24}$$

From Eq (22) it may be seen that the consequence of r being able to take on an infinite number of values is that in a given vernier family there are an infinite number of monotone sequential verniers.

From the relations that have been previously developed the following expressions for a given family of verniers may be written

$$N' = N'_{sv} - N_v \neq rN_v \mp 1 \tag{30}$$

$$N'' = N''_{sv} - N_v = rN_v \mp 1 \tag{31}$$

where $\qquad N_{sp} = N'_{sp} = N''_{sp}$ and $N_v = N'_v = N''_v$

The prime indicates a non-monotone sequential vernier and the double-prime indicates a monotone sequential vernier. The intrafamily relationship between the two types of verniers will now be developed by first assuming a relationship and then showing that such relationship is valid.

An integral relationship is assumed to exist between N'_{sv} and N''_{sv} such that

$$N''_{sv} = t\, N'_{sv} \tag{32}$$

where $t = \pm 1, \pm 2, \pm 3, \ldots$

Substitution of Eq (32) into Eq (31) gives

$$t\, N'_{sv} = N_v + rN_v \mp 1 \tag{33}$$

If $$u \triangleq r + 1 \tag{34}$$

then Eq (33) becomes

$$N'_{sv}\, t - N_v u = \mp 1 \tag{35}$$

where $u = 0, 1, 2, 3, \ldots$

Equation (35) is a linear indeterminate equation with two unknowns, t and u. The factors N'_{sv} and N_v determine a specific non-monotone sequential vernier within the given family. Also, since u is a function of r, the two factors u and N_v determine a specific monotone sequential vernier. Thus, if Eq (35) has a solution such that u is a positive integer, that solution proves the initial assumption, Eq (32); also, the solution gives the integral relationship t that exists between the two types of verniers.

A linear indeterminate (Diophantine) equation with two unknowns in the form

$$ax + by = c \tag{36}$$

has a solution in integers if and only if the greatest common divisor of a and b divides c. If a and b are relatively prime (greatest common divisor of one) and x_o, y_o is a solution, then there are other solutions which are given by

$$x = x_o + bm \qquad y = y_o - am \tag{37}$$

where m is an integer. Proof of these statements may be found in any standard text on number theory (Ref 3:148).

Since N'_{sv} and N_v are relatively prime, Eq (35) meets the above condition and has solutions of the form

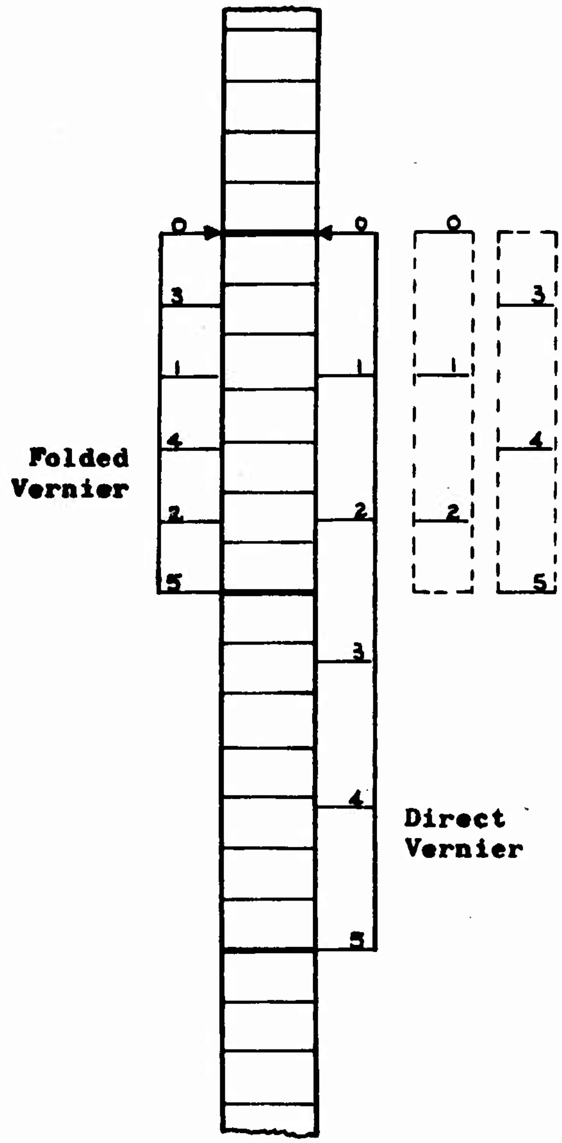
$$t = t_o + mN_v \qquad u = u_o - mN'_{sv} \tag{38}$$

The factor m may take on any integer value; hence, there will be some solution to Eq (35) where u is a positive integer. As a result, for every non-monotone sequential vernier there will be a monotone sequential vernier that is directly related to it. This relationship will be expressed by Eq (32) where the factor t will be determined by Eq (35).

It should be noted that Eq (35) is actually two equations; thus, there will be at least two values of t that will provide integral relationships between the two types of vernier scales. One value of t will relate the folded (non-monotone sequential) vernier to the direct vernier; the other value of t will relate the folded vernier to the retrograde vernier. Figures 8 and 9 illustrate these intrafamily relationships.

A study of Figs. 8 and 9 from a geometric point of view reveals that in order to have a direct correlation between the two types of verniers it is necessary that

$$N_v L''_v = |t| \, N_v L'_v \tag{39}$$



Direct Vernier Parameters

$N_{sp} = 7$ $\quad$ $N_{sv} = 14$

$N_v = 5$ $\quad$ $N = 9$

$r = 2$

Folded Vernier Parameters

$N_{sp} = 7$ $\quad$ $N_{sv} = 7$

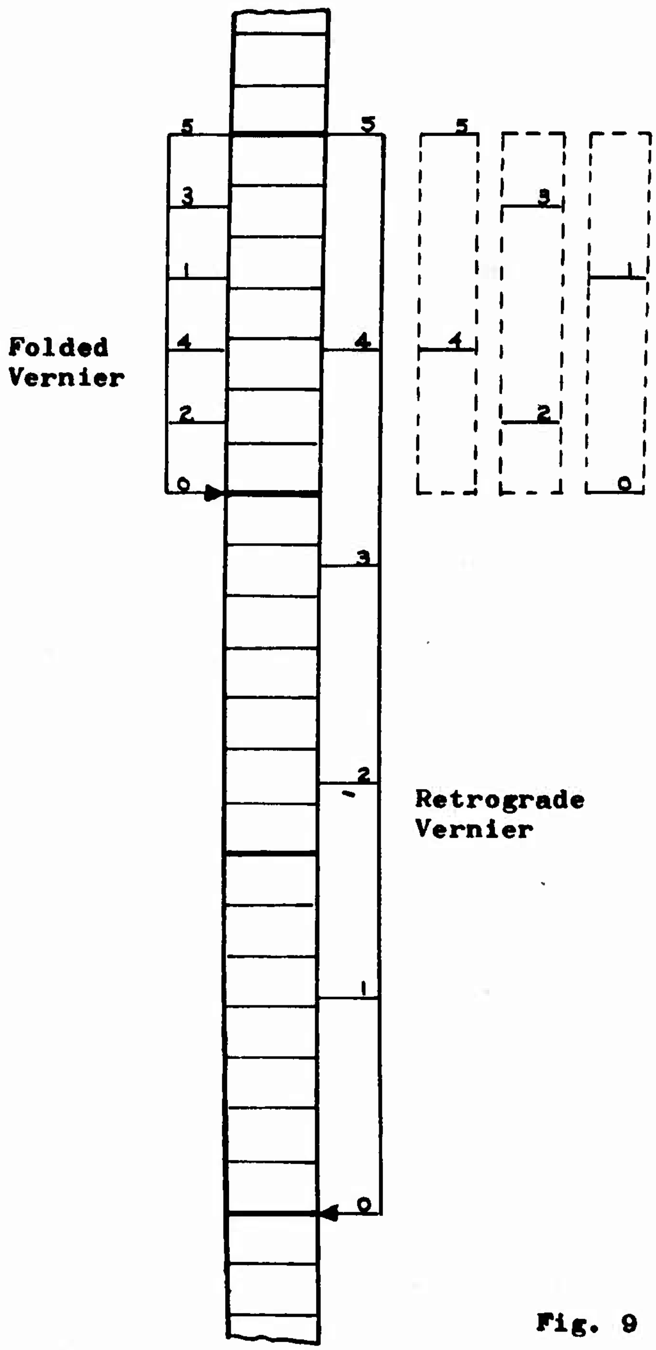
$N_v = 5$ $\quad$ $N = 2$

$t = 2$

Fig. 8

Intrafamily Relationships Between

Direct And Folded Verniers



Retrograde Vernier Parameters

$N_{sp} = 7$ $N_{sv} = 21$

$N_v = 5$ $N = 16$

$r = 3$

Folded Vernier Parameters

$N_{sp} = 7$ $N_{sv} = 7$

$N_v = 5$ $N = 2$

$t = 3$

Fig. 9

Intrafamily Relationships Between

Retrograde And Folded Verniers

This equation shows that a folded vernier may be considered as being constructed from a monotone sequential vernier. This construction is accomplished by dividing the monotone sequential vernier into $|t|$ sections of equal length and then stacking the sections as shown in the Figs. 8 and 9.

Summary of Pertinent Results. The pertinent results from the above development may be summarized as follows:

1. To prevent ambiguity in the scale readings, it is necessary that N_{sv} and N_v be relatively prime.

2. The basic equation describing vernier operation is

$$x = p + \frac{s}{N_{sp}} + \frac{v}{N_{sp}N_v} = \frac{pN_{sp}N_v + sN_v + v}{N_{sp}N_v} \qquad (8)$$

where p, s, and v are the respective readings of the primary, secondary, and vernier scales.

3. The coding of the vernier scale is determined by

$$\left\{v_n\right\} = \left\{q(n)N_v - nN\right\} \qquad (18)$$

where $n = 0,1,2, \ldots, N_v$ and $0 \leq v_n \leq N_v$

4. To construct a monotone sequential vernier it is necessary to satisfy the equation

$$N = rN_v \mp 1 \tag{22}$$

where $r = -1, 0, 1, 2, 3, \ldots$

(the use of the minus sign results in a direct vernier; the plus sign gives a retrograde vernier). If Eq (22) is not satisfied, the resulting device will be a folded (non-monotone sequential) vernier.

5. The least count of a vernier is given by $\dfrac{L_p}{N_{sp} N_v}$.

6. For a given vernier family (same least count) a folded vernier is related to a monotone sequential vernier by

$$N''_{sv} = t\, N'_{sv} \tag{32}$$

The factor t is an integer and is determined by the linear indeterminate equation

$$N'_{sv}\, t - N_v u = \mp 1 \tag{35}$$

where u is a positive integer.

III. Residue Number Systems

As mentioned earlier, residue number systems have characteristics that recommend them for use as computer codes. The development of these novel number systems from congruence theory will be briefly examined in this chapter. Unlike the preceding and succeeding chapters, this chapter presents material that is readily available in other works. The reason it is included in this paper is to furnish some background for the reader who is not familiar with residue numbers. It should be understood, however, that the material presented here is very limited in scope and is developed for specific application in this paper. In particular, the computer applications of these number systems will not be discussed. The reader interested in this aspect of residue numbers is referred to the works of Aiken and Semon (Ref 1) and Garner, et al. (Ref 2).

Residue number systems include both fractional and integer number systems. Though the fractional systems allow greater flexibility with respect to the problem of scaling, residue number systems are more naturally interpreted as integer systems. This paper will consider only the integer interpretation.

Congruences

Though the idea of congruences goes far back into history, it was Gauss who formalized the concept and introduced the terminology and symbolism that are in use today. He defined two integers x and y as being congruent for the modulus m when their difference $x - y$ is divisible by the integer m. Expressed in another way, if the two integers x and y have the same remainder after division by m, they are said to be congruent modulo m. Gauss expressed this definition symbolically by

$$x \equiv y \pmod{m} \tag{40}$$

which is read: x is congruent to y modulo m. The following examples illustrate this concept:

$$25 \equiv 40 \pmod{3} \qquad 25 \equiv 16 \pmod{3}$$

$$25 \equiv 1 \pmod{3} \qquad 25 \equiv -2 \pmod{3}$$

Many of the basic properties of congruences are the same as those of ordinary equalities, and the rules for operating with congruences in many ways resemble those used for combining equations. However, because of more familiarity with them, most

people prefer to work with equations rather than congruences. Aiken and Semon have introduced a notation that satisfies this preference (Ref 1: Chap. I, p. 2). Their notation is based on the fact that a congruence states that the equation

$$x = mq + y \tag{41}$$

is valid for some value of q, where x, y, m, and q are integers. The factors q and y are the quotient and residue (remainder) after division of x by m. If q has such a value that $0 \leq y < m$, then y is said to be the "least positive residue". Using this idea of least positive residue, Aiken and Semon have rewritten Eq (41) as

$$x = m\left[\frac{x}{m}\right] + |x|_m \tag{42}$$

where: $|x|_m$ = least positive residue of x mod m

$\left[\frac{x}{m}\right]$ = largest integer smaller than or equal to x/m

It should be noted that $|x|_m$ is uniquely determined when x is given; however, the converse is not true since $|x|_m$ represents all integers with residue $|x|_m$ after division of x by m. This

relationship between x and $|x|_m$ may be expressed in the language of modern algebra as a many-to-one correspondence of the natural integers x to the least positive residues of x mod m.

__Chinese Remainder Theorem__

In order to have a coherent number system based on residue numbers it is necessary that x and $|x|_m$ be in one-to-one correspondence. There exists a theorem in number theory that gives the necessary and sufficient conditions for obtaining this correspondence for a finite set of natural integers. The first known application of this theorem dates back to about the first century A.D. and is found in a Chinese arithmetic by Sun-Tse. As a result, this theorem is often referred to as the Chinese Remainder Theorem.

The Chinese Remainder Theorem states that if a system of simultaneous congruences is given where the moduli m_i are relatively prime in pairs and

$$M = \prod_{i=1}^{n} m_i \qquad (43)$$

then the set of n least positive residues $|x|_{m_i}$ uniquely determines the integer x in $0 \leq x < M$. The solution for such a system of congruences is given by

$$x \equiv \sum_{i=1}^{n} k_i \hat{m}_i \left| x \right|_{m_i} \pmod{M} \qquad (44)$$

where

$$\hat{m}_i \triangleq \frac{M}{m_i} \qquad (45)$$

$$k_i \hat{m}_i \equiv 1 \pmod{m_i} \qquad 946)$$

The proof of this theorem may be readily found in any standard text on number theory (Ref 3:244-246).

With the notation of Aiken and Semon, the congruences of Eqs (44) and (46) may be transformed into equalities. Equation (44) becomes

$$\left| x \right|_M = \left| \sum_{i=1}^{n} k_i \hat{m}_i \left| x \right|_{m_i} \right|_M \qquad (47)$$

which may be written as

$$x = \sum_{i=1}^{n} k_i \hat{m}_i \left| x \right|_{m_i} - W(x)M \qquad (48)$$

where $W(x)$ is an integral function of x (Ref 3:86). Equation (46) becomes

$$k_i = \left| \frac{1}{\hat{m}_i} \right|_{m_i} \qquad (49)$$

Residue Numbers

From the Chinese Remainder Theorem it is immediately apparent that a number system based on least positive residues can be constructed where

$$x \leftrightarrow (|x|_{m_1}, |x|_{m_2}, \ldots, |x|_{m_n}) \qquad (50)$$

However, in order to have a one-to-one correspondence of x to $(|x|_{m_1}, |x|_{m_2}, \ldots, |x|_{m_n})$ it is necessary that x be bounded. The bounds as stated in the Chinese Remainder Theorem are $0 \leq x < M$. An example of such a number system where $x \leftrightarrow (|x|_2, |x|_3, |x|_5)$ is given in Fig. 10.

For certain applications of residue numbers there appears to be an advantage in having a system where the bounds on x are such that they include numbers other than the least positive residue. To accomplish this change in the range of x, it should be noted that in a residue number system x and $x + nm$, where n is an integer, may be regarded as equivalent. Thus, to obtain the desired range, it is only necessary to redefine W(x) so that $K \leq x < M+K$. An example of a residue number system where $x \leftrightarrow (|x|_2, |x|_3, |x|_5)$ and $-15 \leq x < 15$ is given in Fig. 11.

$$x = \sum_{i=1}^{n} k_i \hat{m}_i |x|_{m_i} - (M+K)\, W(x) \qquad k_i \hat{m}_i \equiv 1 \pmod{m_i}$$

$$\begin{array}{ll} 15\, k_1 \equiv 1 \pmod{2} & k_1 = 1 \\ 10\, k_2 \equiv 1 \pmod{3} & k_2 = 1 \\ 6\, k_3 \equiv 1 \pmod{5} & k_3 = 1 \end{array}$$

$$x = 15\, |x|_2 + 10 |x|_3 + 6 |x|_5 - 30\, W(x)$$

$m_1 = 2$			$m_2 = 3$				$m_3 = 5$		
$0 \leq x < 30$									
x	$\lvert x\rvert_2$	$\lvert x\rvert_3$	$\lvert x\rvert_5$	$W(x)$	x	$\lvert x\rvert_2$	$\lvert x\rvert_3$	$\lvert x\rvert_5$	$W(x)$
0	0	0	0	0	15	1	0	0	0
1	1	1	1	1	16	0	1	1	0
2	0	2	2	1	17	1	2	2	1
3	1	0	3	1	18	0	0	3	0
4	0	1	4	1	19	1	1	4	1
5	1	2	0	1	20	0	2	0	0
6	0	0	1	0	21	1	0	1	0
7	1	1	2	1	22	0	1	2	0
8	0	2	3	1	23	1	2	3	1
9	1	0	4	1	24	0	0	4	0
10	0	1	0	0	25	1	1	0	0
11	1	2	1	1	26	0	2	1	0
12	0	0	2	0	27	1	0	2	0
13	1	1	3	1	28	0	1	3	0
14	0	2	4	1	29	1	2	4	1

Fig. 10

Residue Number System For

$x \longleftrightarrow (|x|_2, |x|_3, |x|_5)$

$0 \leq x < 30$

$$x = \sum_{i=1}^{n} k_i \hat{m}_i |x|_{m_i} - (M+K)\, W(x) \qquad k_i \hat{m}_i \equiv 1 \pmod{m_i}$$

$$15\,k_1 \equiv 1 \pmod 2 \qquad k_1 = 1$$
$$10\,k_2 \equiv 1 \pmod 3 \qquad k_2 = 1$$
$$6\,k_3 \equiv 1 \pmod 5 \qquad k_3 = 1$$

$$x = 15|x|_2 + 10\,|x|_3 + 6\,|x|_5 - 30\,W(x)$$

$m_1 = 2$			$m_2 = 3$				$m_3 = 5$		
$-15 \le x < 15$									
x	$\|x\|_2$	$\|x\|_3$	$\|x\|_5$	$W(x)$	x	$\|x\|_2$	$\|x\|_3$	$\|x\|_5$	$W(x)$
0	0	0	0	0	-15	1	0	0	1
1	1	1	1	1	-14	0	1	1	1
2	0	2	2	1	-13	1	2	2	2
3	1	0	3	1	-12	0	0	3	1
4	0	1	4	1	-11	1	1	4	2
5	1	2	0	1	-10	0	2	0	1
6	0	0	1	0	-9	1	0	1	1
7	1	1	2	1	-8	0	1	2	1
8	0	2	3	1	-7	1	2	3	2
9	1	0	4	1	-6	0	0	4	1
10	0	1	0	0	-5	1	1	0	1
11	1	2	1	1	-4	0	2	1	1
12	0	0	2	0	-3	1	0	2	1
13	1	1	3	1	-2	0	1	3	1
14	0	2	4	1	-1	1	2	4	2

Fig. 11

Residue Number System For

$x \longleftrightarrow (|x|_2, |x|_3, |x|_5)$

$-15 \le x < 15$

IV. Two Scale Residue Verniers

The preceding chapters have presented a theory of conventional vernier devices and have briefly explained the concept of residue numbers. It will now be shown that these seemingly unrelated subjects are, in fact, very closely related. Also, it will be shown that a conventional vernier device, with slight modification, may be used to provide direct conversion from decimal numbers to residue numbers. The author has chosen to refer to the modified device as a "residue vernier".

The relationship between verniers and residue numbers will first be developed for the simplest case - a vernier device consisting of two scales and a residue number system based on two moduli. The theory resulting from this case will then be used as the framework for developing a general theory for multi-scale residue verniers.

Definitions and Conditions

For the development of the theory of conventional verniers it was necessary to consider the main scale as being made-up of two scales, a primary scale and a secondary scale. Since the function of residue verniers is not to measure length, the viewpoint of two scales is not required. In fact, it will be conceptually advantageous to consider the main scale as being a

cyclic scale having exactly the same characteristics as the vernier scale. To decrease the possibility of confusion between these different interpretations of the main scale, the main scale of the residue vernier will be referred to as the "static" scale.

In order to have a number system based on integers, it is essential that the relative displacement between the static and vernier scales be expressible in terms of integers. To provide this relation a basic length different from that used in Chapter II is needed. In that chapter L_p, the length of one primary scale division, was basic. From Eq (8b) it may be seen that if the basic length is considered to be the least count, $\frac{L_p}{N_{sp}N_v}$, rather than L_p, then x will be an integer. Therefore, the unit length based on the least count will be used for residue verniers.

With the above conditions in mind, the following definitions are given:

N_s = number of intervals on the static scale per static scale cycle

N_v = number of intervals on the vernier scale per vernier scale cycle

N_{sv} = number of intervals on the static scale per vernier scale cycle

L_x = unit length (least count)

L_s = length of a static scale interval

L_v = length of a vernier scale interval

L_S = length of a static scale cycle

L_V = length of a vernier scale cycle

It should be noted that N_s and L_S for a residue vernier equal respectively N_{sp} and L_p for a conventional vernier. Also, the unit length for a two-scale residue vernier equals the least count of a conventional vernier:

$$L_x = \frac{L_p}{N_{sp} N_v} = \frac{L_S}{N_s N_v} \tag{51}$$

It must be emphasized that Eq (51) only holds for two-scale residue verniers. Later development will show that a different relation is required for multi-scale devices.

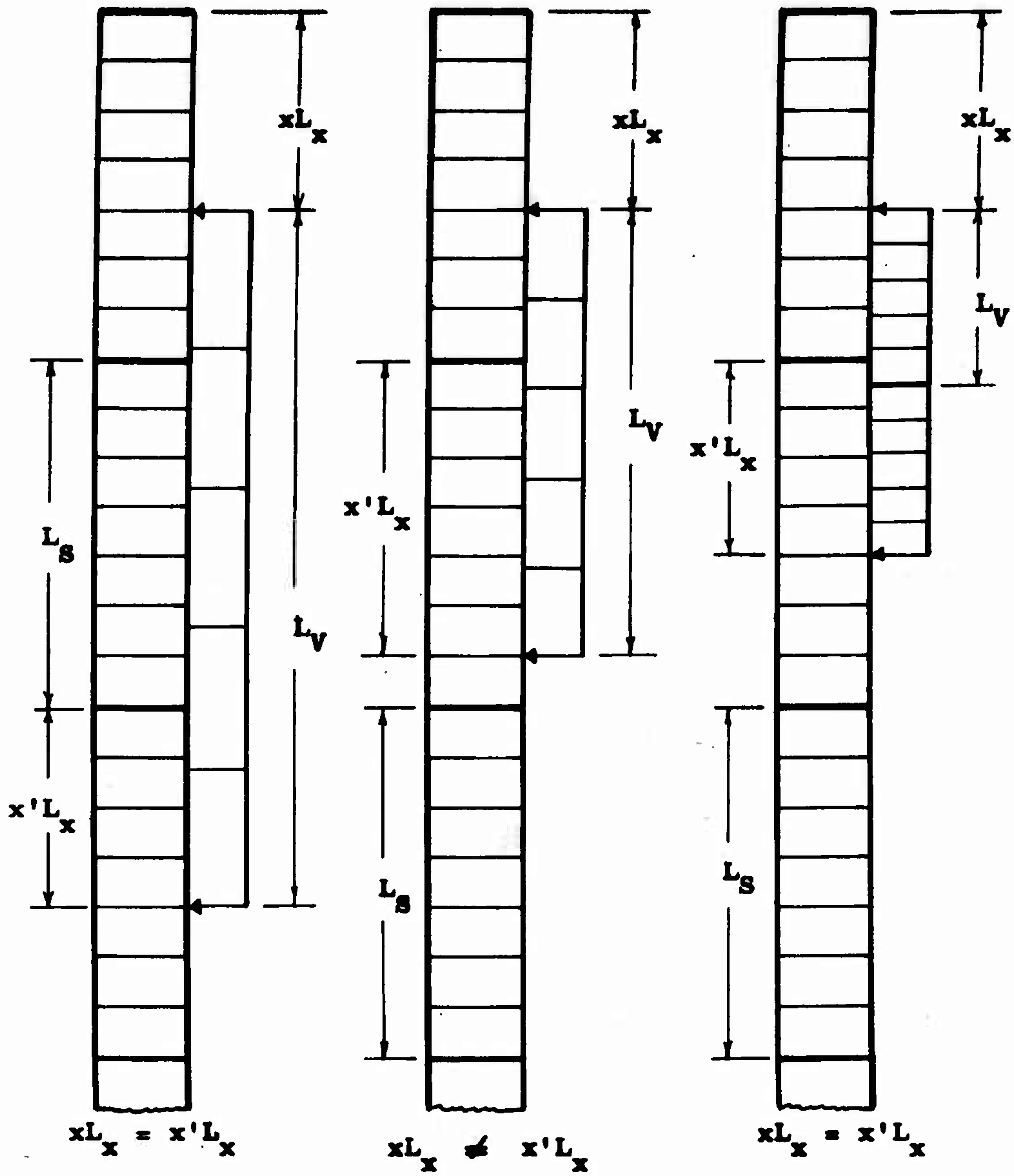
The factors N_v, N_{sv}, L_s, and L_v have the same meanings as they had for conventional verniers. Unlike conventional verniers, however, the principal requirement for the design of residue verniers is that N_s and N_v be relatively prime. This is an important change, and the reader should keep it in mind so that he may better understand the subsequent development. The reason for this change, though not apparent now, will become obvious later on.

Another innovation is based on the fact that the choice of the line to serve as the fiducial is not completely axiomatic. This is true because N_{sv} is stipulated to be an integer. This stipulation results in the lines on the vernier scale a distance L_v apart having simultaneous coincidence (e.g., Figs. 5, 6, and 7). As a result, the line a distance L_v from the fiducial may as easily be coded zero as N_v. Hence, the range for v_n as given by Eq (18) for conventional verniers may be changed to

$$0 \leq v_n < N_v \tag{52}$$

With this condition it may be seen that the two lines bounding a vernier scale cycle may perform the function of the fiducial equally well. This concept has little practical value for conventional verniers; for residue verniers, however, it provides a key to the understanding of their operation.

In Fig. 12 three linear verniers are illustrated, each with two fiducials. The properties of interest are the distances between the beginning of a static scale cycle and the fiducials. Examination of the three verniers shows that the two distances, xL_x and $x'L_x$, are not necessarily equal. In order for these distances to be equal, it is necessary that the length of the vernier scale cycle L_v and the length of the



(a) $L_V = 2L_S$ (b) $bL_V \neq aL_S$ (c) $L_S = 2L_V$

Fig. 12

Verniers With Two Fiducials

static scale cycle L_S have an integral relationship such that

$$bL_V = aL_S \tag{53}$$

where $a = 1,2,3, \ldots$ and $b = 1,2,3, \ldots$

For the development of the theory of two-scale residue verniers the factor b will be assumed to be one, so

$$L_V = aL_S \tag{54}$$

In the subsequent development of the theory of multi-scale devices, b will be allowed to take on other values.

From the definitions given at the beginning of this section and from Eq (51), it may be seen that

$$L_V = N_v L_v \tag{55}$$

$$L_S = N_s L_s \tag{56}$$

$$N_{sv} L_s = N_v L_v \tag{57}$$

These relations applied to Eq (54) give

$$N_{sv} = a\, N_s \tag{58}$$

Thus, when a vernier device's parameters are such that Eq (58) is satisfied, the two fiducials will give equal readings ($x = x'$). As will be seen later, this equality is a necessary condition for the proper operation of residue verniers.

In the above development there has been no explicit limitation on the factor a other than it has to be a positive integer. There is, however, an implicit condition that a must satisfy which does place limitations on the values it may assume. As was found in Chapter II for conventional verniers, it is necessary for two-scale residue verniers that N_v and N_{sv} be relatively prime so as to prevent ambiguity of scale readings. The standard notation used to express the idea of relative primeness is

$$(N_v, N_{sv}) = 1 \qquad (59)$$

which is read: the greatest common divisor of N_v and N_{sv} is one (Ref 4:47). A vernier device whose parameters satisfy the conditions of both Eqs (58) and (59) has

$$(N_v, a\,N_s) = 1 \qquad (60)$$

Thus, a may no longer be any positive integer, but may assume only those values which make the product $a\,N_s$ relatively

prime to N_v (e.g., if $a = N_v$, then Eq (60) doesn't hold).

In any standard text on number theory it is shown that when a number is relatively prime to each of several numbers, then that number is also relatively prime to the product of the several numbers (Ref 4:44). As previously mentioned, residue verniers require that

$$(N_v, N_s) = 1 \tag{61}$$

Given this relation, a sufficient condition for satisfying Eq (60) is

$$(N_v, a) = 1 \tag{62}$$

Thus, if a vernier device's parameters (a, N_v, and N_s) satisfy the conditions of Eqs (61) and (62), this will be sufficient for Eq (60) to be satisfied, and it follows from Eq (58) that $x = x'$.

With the above definitions and conditions, it is now possible to show the relationship between vernier devices and residue numbers.

The Vernier Scale

A distance equation based on the inside dimensions of

Fig. 13 may be written such that

$$xL_x = hL_s + mL_s - nL_v \tag{63}$$

where, by the definition of L_x, the factor x is an integer. Equations (51), (54), (55), and (56) may be appropriately combined to obtain

$$L_s = \frac{L_S}{N_s} = \frac{N_s N_v L_x}{N_s} = N_v L_x \tag{64}$$

$$L_v = \frac{L_V}{N_v} = \frac{aL_S}{N_v} = a\, N_s L_x \tag{65}$$

Substitution of Eqs (64) and (65) into Eq (63) gives

$$x = hN_v + (mN_v - naN_s) \tag{66}$$

Taking the lead from Chapter II, the right-hand term is defined as v_n, the numerical code given a line on the vernier scale. Thus

$$v_n = mN_v - naN_s = q(n)N_v - anN \tag{67}$$

where: $$n = 0,1,2,\ \ldots,\ N_v \tag{67}$$

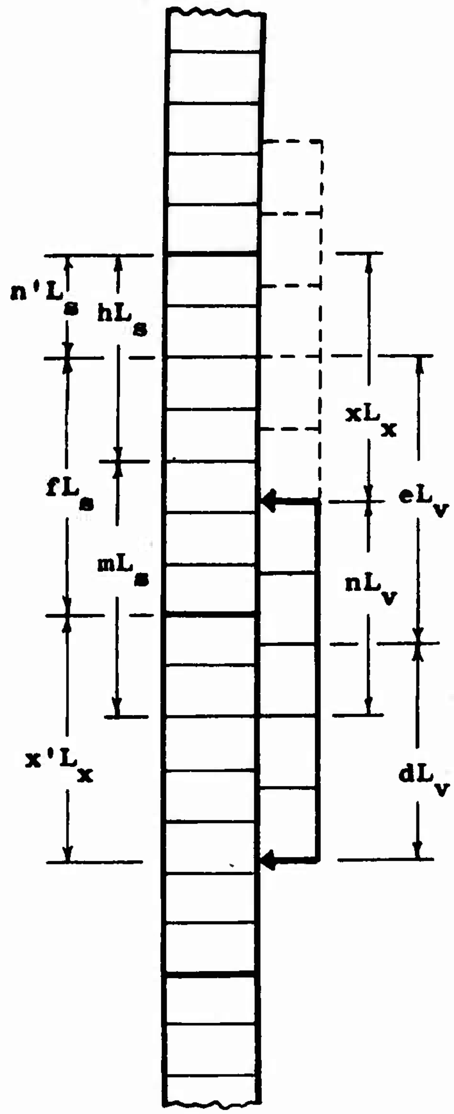


Fig. 13

Parameters For Vernier Device

With Two Fiducials

$$q(n) = m - an \tag{69}$$

$$N = N_s - N_v \tag{70}$$

and from Eq (52) $\quad 0 \leq v_n < N_v \qquad (52)$

With the above definition, Eq (66) becomes

$$v_n = x - N_v F(x) \tag{71}$$

where $\quad 0 \leq v_n < N_v \qquad h = F(x)$

From Chapter III it is possible to write

$$|x|_{N_v} = x - N_v f(x) \tag{72}$$

where $\quad 0 \leq |x|_{N_v} < N_v$

Comparison of Eqs (71) and (72) shows that v_n satisfies the conditions for the least positive residue of $x \bmod N_v$; hence,

$$v_n = |x|_{N_v} \tag{73}$$

<u>The Static Scale</u>

An equivalent relation to that given in Eq (73) may be developed for the static scale by again writing a distance

equation. This equation, however, requires the use of the outside dimensions in Fig. 13. Hence,

$$x'L_x = dL_v + eL_v - fL_s \tag{74}$$

Substitution of Eqs (64) and (65) into Eq (74) gives

$$x' = daN_s + eaN_s - fN_v \tag{75}$$

From Fig. 13 it may be seen that the factor f determines the number of static scale intervals from the beginning of the cycle much in the same way as n determines the number of vernier scale intervals. However, there is one important difference. The factor f measures the intervals in a direction opposite to the direction in which n determines vernier scale intervals. From a practical viewpoint, it would be more convenient for the two coding sequences to code their respective scales in the same direction. From Fig. 13 it may be seen that

$$n'L_s = N_sL_s - fL_s \tag{76}$$

where $n'L_s$ is the distance from the beginning of the static scale cycle measured in the same direction as n measures vernier scale intervals. Hence,

$$f = N_s - n' \tag{77}$$

To accomplish the desired change, Eq (77) may be substituted into Eq (75), which gives

$$x' = daN_s + (ea - N_v + n')\, N_s - n'\, (N_s - N_v) \tag{78}$$

If
$$G(x') = da \tag{79}$$

$$q(n') = ea - N_v + n \tag{80}$$

$$N = N_s - N_v \tag{70}$$

then
$$x' = G(x')N_s + \left[q(n')\, N_s - n'N\right] \tag{81}$$

The same arguments used in Chapter II to develop the vernier scale coding sequence from a distance equation may be applied to Eq (81). If $s_{n'}$ is defined as the numerical value given a static scale line n' intervals from the beginning of the cycle, then

$$\left\{s_{n'}\right\} = \left\{q(n')\, N_s - n'N\right\} \tag{82}$$

where $n' = 0,1,2, \ldots ,N_s$ and $0 \leq s_{n'} < N_s$

The substitution of Eq (82) into Eq (81) gives

$$s_{n'} = x' - N_s G(x') \tag{83}$$

Thus, from residue number theory it may be seen that

$$s_{n'} = |x'|_{N_s} \tag{84}$$

The Two-Scale Residue Vernier

It has been previously shown that for a vernier device with parameters which satisfy Eqs (61) and (62) the relation $x = x'$ will hold. For such devices Eqs (71) and (73) and Eqs (83) and (84) show that when there is coincidence of scales

$$x = |x|_{N_v} + N_v F(x) \tag{85}$$

$$x = |x|_{N_s} + N_s G(x) \tag{86}$$

where $|x|_{N_v}$ and $|x|_{N_s}$ are the readings of the scales at that coincidence.

The Chinese Remainder Theorem states that a set of equations like Eqs (85) and (86) uniquely determine x in $0 \leq x < N_s N_v$ if N_s and N_v are relatively prime. Thus, the

scale readings of a vernier device constructed with appropriate parameters and properly coded will at coincidence provide the least positive residues of x modulo N_s and modulo N_v:

$$x \longleftrightarrow (|x|_{N_s}, |x|_{N_v}) \qquad (87)$$

where $$0 \leq x < N_s N_v$$

The sufficient conditions and requirements for the design and construction of a two-scale residue vernier given a unit length L_x, the relation $b = 1$, and the parameters a, N_v, and N_s are as follows:

1. $$(N_s, N_v) = 1 \qquad (61)$$

2. $$(N_v, a) = 1 \qquad (62)$$

where $a = 1, 2, 3, \ldots$ and $N_{sv} = aNs$ (58)

3. $$L_s = N_v L_x \qquad (64)$$

4. $$L_v = N_{sv} L_x = aN_s L_x \qquad (65)$$

5. $$\{v_n\} = \{q(n)\, N_v - anN\} \qquad (67)$$

where $n = 0,1,2, \ldots, N_v$ and $0 \leq v_n < N_v$

6. $$\{s_{n'}\} = \{q(n')\, N_s - n'N\} \qquad (82)$$

where $n' = 0,1,2, \ldots, N_s$ and $0 \leq s_{n'} < N_s$

The first two relations establish the sufficient conditions for the design of a two-scale residue vernier; the second two relations determine the physical structure of the device; and, the last two relations provide the coding sequences for the static and vernier scales. All of the relations given above were developed with the assumption that the factor b in Eq (53) was equal to one.

In Figs. 14, 15, and 16 some examples of residue verniers are given where $a = 1$, $b = 1$. To better visualize the operation of the devices shown in these figures, it is suggested that the reader transfer the vernier scale to the edge of a piece of paper. The paper may then be slid along the static scale to obtain any desired coincidence. A reference scale indicating integral multiples of the unit length is placed next to the static scale in each figure for the reader's convenience. Also, a residue number conversion table is given in Fig. 17 for easy reference.

In addition to the operation of residue verniers there are some other precepts that may be obtained from these figures. First, when $a = 1$ $(b = 1)$, the factor N determines the type of scale - direct, retrograde, or folded - for residue verniers in the same manner as it did for conventional verniers. Second, the length of the static scale does not need to be greater than twice the length of the vernier scale in order to provide all the residue numbers for which x is unique.

Figure 18 presents a different constructional approach for residue verniers. In this figure the linear vernier illustrated in Fig. 15 is re-presented as a circular residue vernier. In order to visualize its operation the shaded circle in the center can be considered as a disk mounted on a flat plate in such a way that it is free to rotate. Hence, the disk serves as the vernier scale and the plate serves as the static scale. As in Figs. 14, 15, and 16, a scale indicating the integral multiples of the unit length is placed next to the static scale for convenient reference. The main difference between the circular and linear devices, besides the obvious physical difference, is that the circular device requires a static scale only of a length equal to that of the vernier scale.

The Multi-Cycle Residue Vernier

When $a \neq 1$ a slightly different configuration from these shown in Figs. 14, 15, 16, and 18 results. In each of those

GIVEN PARAMETERS:

$L_x = 1/8''$ $N_s = 5$ $N_v = 6$ $a = 1$

REQUIREMENTS:

1. $(N_s, N_v) = (5, 6) = 1$
2. $(N_v, a) = (5, 1) = 1; N_{sv} = N_s = 5$
3. $L_s = 6(1/8'') = 3/4''$
4. $L_v = 5(1/8'') = 5/8''$

5.

$N_v = 6$ $N = -1$			
n	$q(n)N_v - anN$	v_n	$q(n)$
0	$6q(0)+0$	0	0
1	$6q(1)+1$	1	0
2	$6q(2)+2$	2	0
3	$6q(3)+3$	3	0
4	$6q(4)+4$	4	0
5	$6q(5)+5$	5	0
6	$6q(6)+6$	0	-1

6.

$N_s = 5$ $N = -1$			
n	$q(n')N_s - n'N$	$s_{n'}$	$q(n')$
0	$5q(0)+0$	0	0
1	$5q(1)+1$	1	0
2	$5q(2)+2$	2	0
3	$5q(3)+3$	3	0
4	$5q(4)+4$	4	0
5	$5q(5)+5$	0	-1

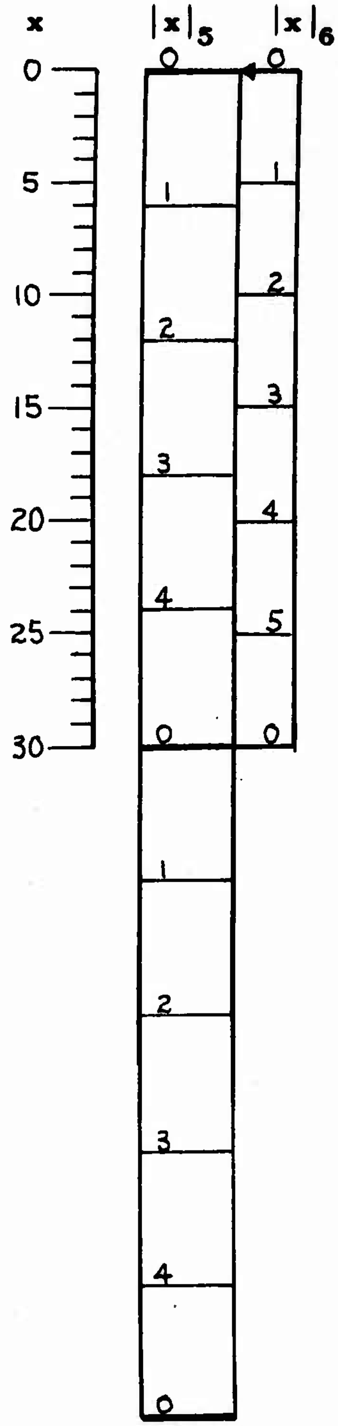
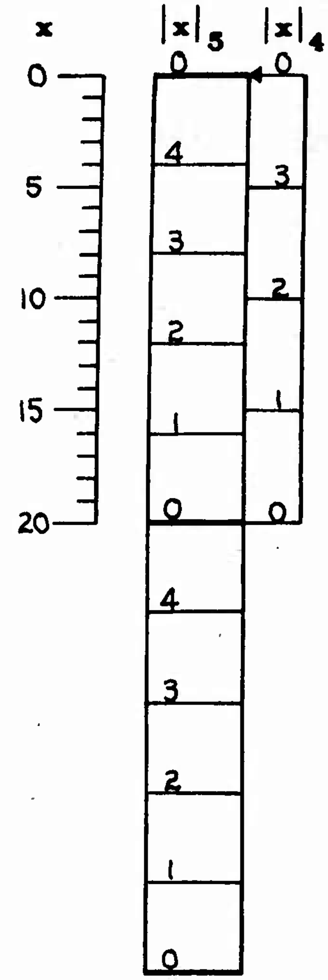


Fig. 14
Direct Residue Vernier

GIVEN PARAMETERS:

$L_x = 1/8''$ $N_s = 5$ $N_v = 4$ $a = 1$



REQUIREMENTS:

1. $(N_s, N_v) = (5, 4) = 1$
2. $(N_v, a) = (5, 1) = 1$; $N_{sv} = N_s = 5$
3. $L_s = 4(1/8'') = 1/2''$
4. $L_v = 5(1/8'') = 5/8''$
5.

$N_v = 4$ $N = 1$			
n	$q(n)N_v - anN$	v_n	q(n)
0	4q(0)-0	0	0
1	4q(1)-1	3	1
2	4q(2)-2	2	1
3	4q(3)-3	1	1
4	4q(4)-4	0	1

$N_s = 5$ $N = 1$			
n'	$q(n')N_s - n'N$	$s_{n'}$	q(n')
0	5q(0)-0	0	0
1	5q(1)-1	4	1
2	5q(2)-2	3	1
3	5q(3)-3	2	1
4	5q(4)-4	1	1
5	5q(5)-5	0	1

Fig. 15
Retrograde Residue Vernier

GIVEN PARAMETERS:

$L_x = 1/8''$ $\quad N_s = 5 \quad N_v = 7 \quad a = 1$

REQUIREMENTS:

1. $(N_s, N_v) = (5, 7) = 1$
2. $(N_v, a) = (5, 1) = 1$; $N_{sv} = N_s = 5$
3. $L_s = 7(1/8'') = 7/8''$
4. $L_v = 5(1/8'') = 5/8''$

5.

$N_v = 7$ $\quad N = -2$			
n	$q(n)N_v - anN$	v_n	$q(n)$
0	7q(0)+0	0	0
1	7q(1)+2	2	0
2	7q(2)+4	4	0
3	7q(3)+6	6	0
4	7q(4)+8	1	-1
5	7q(5)+10	3	-1
6	7q(6)+12	5	-1
7	7q(7)+14	0	-2

6.

$N = 5$ $\quad N = -2$			
n'	$q(n')N_s - n'N$	$s_{n'}$	$q(n')$
0	5q(0)+0	0	0
1	5q(1)+2	2	0
2	5q(2)+4	4	0
3	5q(3)+6	1	-1
4	5q(4)+8	3	-1
5	5q(5)+10	0	-2

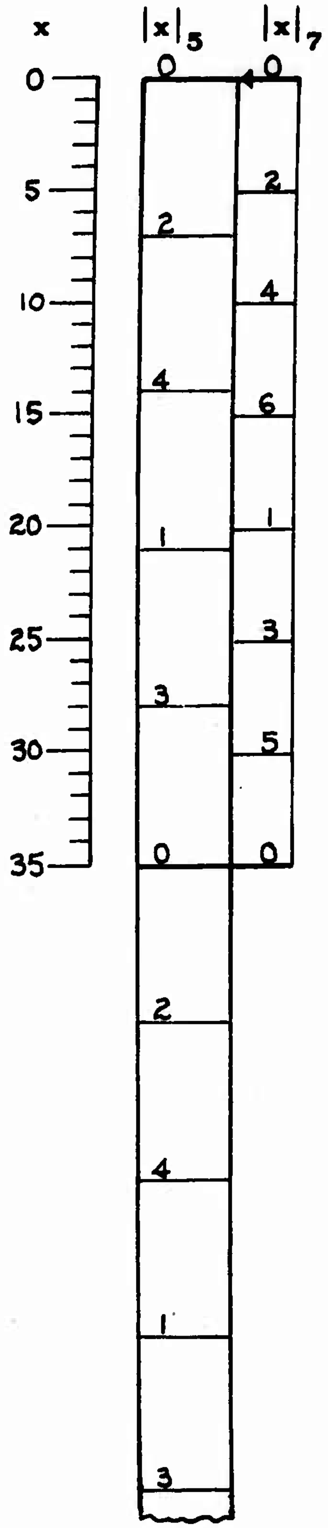


Fig. 16
Folded Residue Vernier

x	$\lvert x \rvert_5$	$\lvert x \rvert_4$	$\lvert x \rvert_6$	$\lvert x \rvert_7$
0	0	0	0	0
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3
4	4	0	4	4
5	0	1	5	5
6	1	2	0	6
7	2	3	1	0
8	3	0	2	1
9	4	1	3	2
10	0	2	4	3
11	1	3	5	4
12	2	0	0	5
13	3	1	1	6
14	4	2	2	0
15	0	3	3	1
16	1	0	4	2
17	2	1	5	3
18	3	2	0	4
19	4	3	1	5
20	0		2	6
21	1		3	0
22	2		4	1
23	3		5	2
24	4		0	3
25	0		1	4
26	1		2	5
27	2		3	6
28	3		4	0
29	4		5	1
30	0			2
31	1			3
32	2			4
33	3			5
34	4			6

Fig. 17
Residue Number Conversion Table

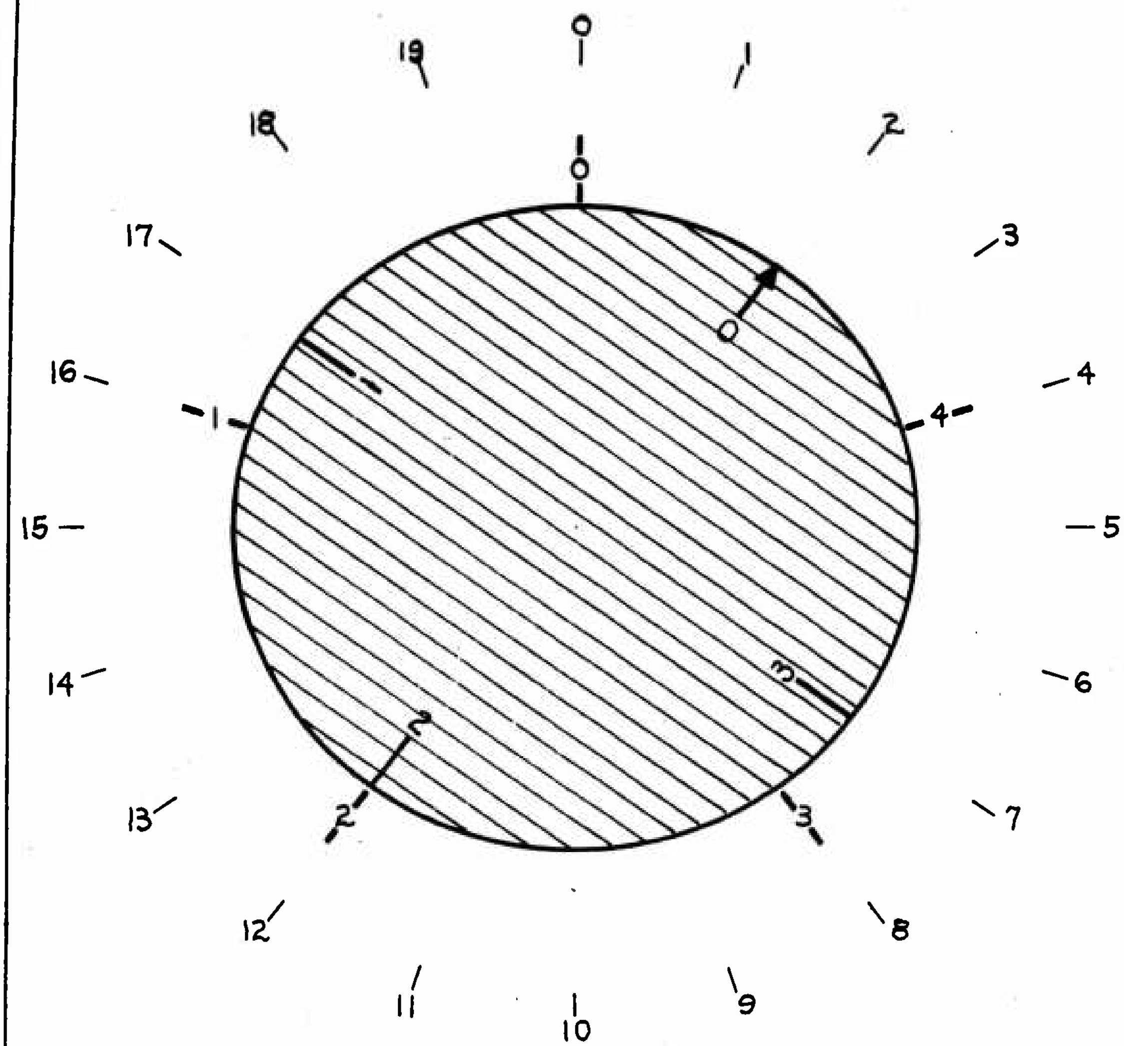


Fig. 18

Circular Retrograde

Residue Vernier

$N_s = 5 \quad N_v = 4 \quad a = 1$

figures it may be seen that the length of the vernier scale cycle is equal to the length of the reference scale which gives a complete cycle of the values of x. Examples of devices for which $a \neq 1$ are given in Figs. 19 and 20. In these figures the length of the vernier scale cycle is equal to $\underline{a}$ times the length of the reference scale cycle. Since it takes $\underline{a}$ reference scale cycles to equal the length of one vernier scale cycle, devices where $a \neq 1$ are referred to as "multi-cycle" residue verniers.

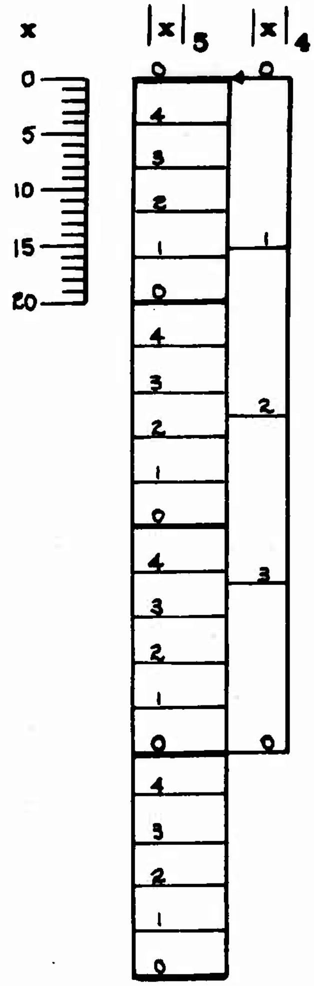
Examination of Figs. 19 and 20 and of Eq (67) reveals that the parameter N no longer completely determines the scale type for the vernier scale. To obtain monotone code sequences on the vernier scale when $a \neq 1$, Eq (22) of Chapter II must be modified to

$$aN = rN_v \mp 1 \qquad (88)$$

As before, the minus sign gives a direct scale, and the plus sign gives a retrograde scale. If the device parameters are such that Eq (88) doesn't hold, then the scale will be of the folded type. It has been previously shown that the ratio of the scale lengths needed to provide uniqueness approaches two when $a = 1$. From these figures it may be seen that as a increases this ratio of scale lengths approaches one. Thus, if

GIVEN PARAMETERS:

$L_x = 1/16"$ $N_s = 5$ $N_v = 4$ $a = 3$



REQUIREMENTS:

1. $(N_s, N_v) = (5, 4) = 1$
2. $(N_v, a) = (4, 3) = 1;\ N_{sv} = aN_s = 15$
3. $L_s = 4(1/16") = 1/4"$
4. $L_v = 15\ (1/16") = 15/16"$

5.

$N_v = 4$	$N = 1$		
n	$q(n)N_v - anN$	v_n	q(n)
0	4q(0)-0	0	0
1	4q(1)-3	1	1
2	4q(2)-6	2	2
3	4q(3)-9	3	3
4	4q(4)-12	0	4

6.

$N_s = 5$	$N = 1$		
n'	$q(n')N_s - n'N$	$s_{n'}$	q(n')
0	5q(0)-0	0	0
1	5q(1)-1	4	0
2	5q(2)-2	3	0
3	5q(3)-3	2	0
4	5q(4)-4	1	0
5	5q(5)-5	0	1

Fig. 19

Three-Cycle Residue Vernier

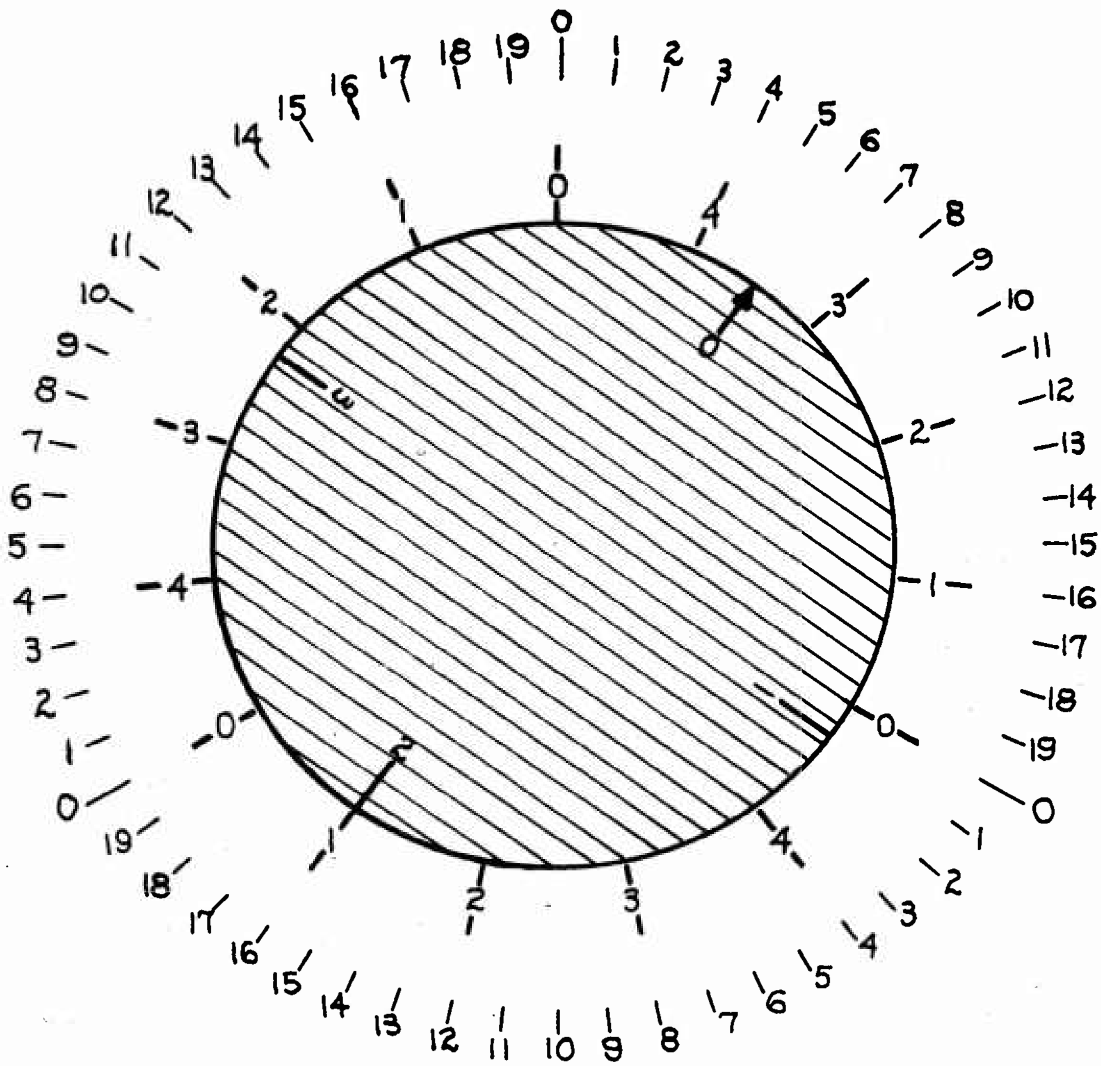


Fig. 20

Circular Three-Cycle

Residue Vernier

$N_s = 5 \quad N_v = 4 \quad a = 3$

N_V = total number of vernier scale cycles required to provide uniqueness

N_S = total number of static scale cycles required to provide uniqueness

then for a linear residue vernier

$$1 < \frac{N_S L_S}{N_V L_V} < 2 \qquad (89)$$

V. Multi-Scale Residue Verniers

The Chinese Remainder Theorem shows that the range in which x has a one-to-one correspondence in a residue number system is determined by the product of the moduli, providing the moduli are relatively prime in pairs. Thus, to extend the range of this correspondence it is necessary either to increase the value of the moduli or to increase the number of the moduli. The previous development has shown that increasing the value of the moduli of a number system associated with a residue vernier presents no theoretical problems (there could be some practical difficulties due to increasing dimensions). The following development will pertain to the latter method of increasing the range of one-to-one correspondence, that of increasing the number of moduli.

To increase the number of moduli of a number system used with a residue vernier necessitates the addition of scales to the device. This addition of scales may be accomplished in three ways: the addition of static scales, the addition of vernier scales, or the addition of both types of scales. For convenience, each of these three methods will be treated separately.

Multi-Static Scale Device

If several two-scale residue verniers are constructed so that they have equal unit lengths and the same vernier scale parameters, it is apparent that their static scales could be placed together and the device would function properly. This concept of a multi-static scale device being a composite structure of several two-scale devices will be used in the following development. The advantage of this approach is that the development problem is reduced to only determining what, if any, constraints are placed on the two-scale device relationships as a result of attaching additional scales.

Device Parameters. For a multi-static scale device with a common unit length and with one vernier scale, the factors L_x, L_V, L_v, and N_v have the same meaning as given them in Chapter IV. The static scale parameters, however, require some modification since there are several scales to consider. The change required is the addition of a sub-subscript which identifies the pertinent scale, i.e., N_{s_j}, $(N_{sv})_j$, L_{s_j}, and L_{S_j} where $j = 1, 2, \ldots, m$.

When the unit length is common and each static scale operates with a common vernier scale, it follows from Eq (64) that

$$N_v L_x = L_{s_j} = L_s \qquad (90)$$

Also, it follows from Eqs (57) and (90) that

$$(N_{sv})_j = \frac{N_v L_v}{L_{s_j}} = \frac{N_v L_v}{L_s} = N_{sv} \qquad (91)$$

Thus, the parameters N_{sv} and L_s are common for all the static scales. This fact could have also been derived from practical considerations since a residue vernier requires coincidence of scales for proper operation.

It was shown in Chapter IV that residue verniers require $x = x'$. From Eq (53) of that chapter it may be seen that the condition needed to satisfy this requirement for multi-static scale devices is

$$bL_v = a_j L_{s_j} \qquad (92)$$

For multi-static scale devices b will again be assumed to equal one. Hence, Eq (92) may be reduced to

$$N_{sv} = a_j N_{s_j} \qquad (93)$$

where the implicit restrictions on a_j result in

$$(a_j , N_v) = 1 \qquad (94)$$

If $$(N_v, N_{s_j}) = 1 \qquad (95)$$

then from the theorem in Chapter IV concerning relative prime numbers a sufficient condition for satisfying Eqs (93) and (94) is

$$a_j = \frac{C_v S}{N_{s_j}} \tag{96}$$

where

$$S \triangleq \prod_{j=1}^{m} N_{s_j} \tag{97}$$

and C_v is defined as a positive integer which satisfies

$$(C_v, N_v) = 1 \tag{98}$$

Substitution of Eq (96) into Eq (95) gives

$$N_{sv} = C_v S \tag{99}$$

From Eqs (57), (90), and (99), it may be seen that

$$L_v = \frac{N_{sv} L_s}{N_s} = C_v S L_x \tag{100}$$

<u>Coding</u> <u>Sequences</u>. The above relations determine the constraints placed on the physical structure of a one vernier scale, multi-static scale residue vernier. It is now necessary to determine what changes, if any, need to be made to the coding sequences developed for the two-scale device.

Since the multi-scale device may be viewed as a composite structure of several two-scale devices, it is permissible to

use the distance equations developed in Chapter IV. Hence,

$$xL_x = hL_s + mL_s - nL_v \qquad (63)$$

Substitution of Eqs (90) and (100) into Eq (63) gives

$$x = hN_v + (mN_v - nC_vS) \qquad (101)$$

Thus, from prior logic the coding sequence for the vernier scale is

$$\{v_n\} = \{q(n)\ N_v - nC_vN\} \qquad (102)$$

where: $n = 0,1,2, \ldots, N_v$ $\qquad 0 \leq v_n < N_v$

$q(n) = m - nC_v$ $\qquad N = S - N_v$

The other distance equation from Chapter IV gives

$$x'L_x = d_jL_v + e_jL_v - f_jL_s \qquad (74)$$

Substitution of Eqs (77), (90), and (100), into Eq (74) gives

$$x' = d_jC_vS + \left[\left(\frac{e_jC_vS}{N_{s_j}} - N_v + \frac{n_jS}{N_{s_j}}\right) N_{s_j} - n_j\ (s - N_v)\right] \qquad (103)$$

(the prime has been removed from n' since n and n' have the same basic meaning). Thus, the coding sequence for the static scales becomes

$$\left\{ s_{n_j} \right\} = \left\{ q(n_j)\, N_{s_j} - n_j N \right\} \qquad (104)$$

where: $n_j = 0, 1, 2, \ldots, N_{s_j}$ $\qquad 0 \leq s_{n_j} < N_{s_j}$

$$q(n_j) = \frac{e_j C_v S}{N_{s_j}} - N_v + \frac{n_j S}{N_{s_j}} \qquad N = S - N_v$$

From arguments similar to those used in Chapter IV, it follows from Eqs (101) and (102) and Eqs (103) and (104) that at a coincidence of scales for a multi-static scale residue vernier

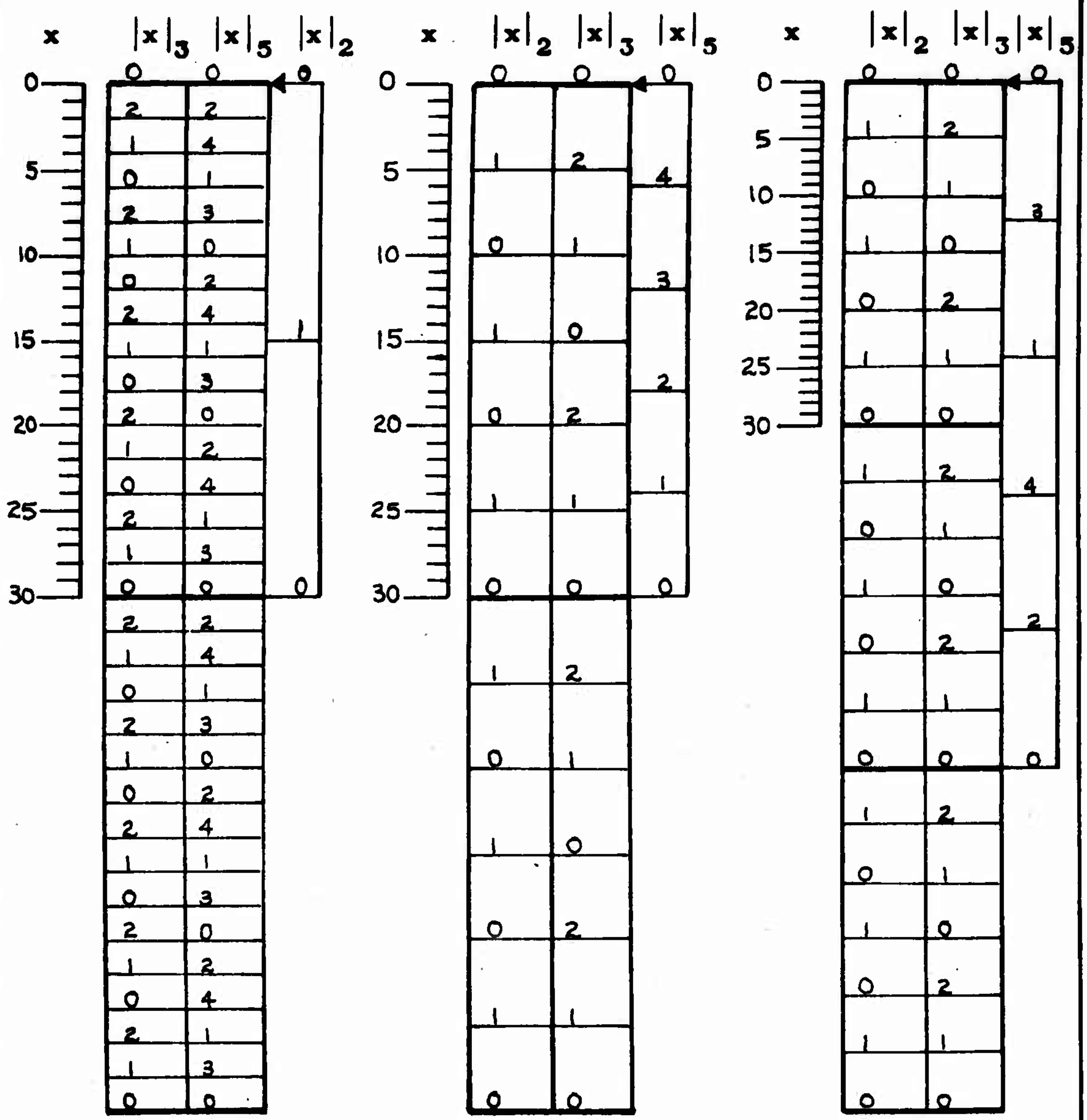
$$x = |x|_{N_v} + N_v F(x) \qquad (105)$$

$$x = |x|_{N_{s_j}} + N_{s_j} G_j(x) \qquad (106)$$

where : $\qquad F(x) = h \qquad G_j(x) = \dfrac{dC_v S}{N_{s_j}}$

Hence, by the Chinese Remainder Theorem

$$x \longleftrightarrow \left(|x|_{N_v},\ |x|_{N_{s_1}}, |x|_{N_{s_2}},\ \ldots,\ |x|_{N_{s_m}} \right) \qquad (107)$$



(a) $C_v = 1$, $N_v = 2$
$N_{s_1} = 3$, $N_{s_2} = 5$

(b) $C_v = 1$, $N_v = 5$
$N_{s_1} = 2$, $N_{s_2} = 3$

(c) $C_v = 2$, $N_v = 5$
$N_{s_1} = 2$, $N_{s_1} = 3$

Fig. 21
Multi-Static Scale
Residue Verniers

where $$0 \leq x < N_v S$$

Figure 21 gives some simple examples of one vernier scale, multi-static scale residue verniers. It may be seen in these figures that the parameter C_v determines the cyclic nature of the reference scale with respect to the vernier scale, just as the factor a did for two-scale devices. Also, comparison of Fig. 21(a) and Fig. 21(b) indicates that from a practical viewpoint it may be advantageous to let N_v be the largest moduli of the number system used since this allows fewer intervals on the static scale.

Multi-Vernier Scale Devices

Device Parameters. In the preceding section it was convenient to consider the multi-static scale device as a marriage of several two-scale devices. This concept will also be used for developing the relationships for a multi-vernier scale device. However, there is a modification that must be made. In all the previous developments the integer N_{sv} has been used as the connecting link between the static and vernier scales. For multi-vernier scale devices this parameter has little meaning since there are several vernier scales to be considered. Hence, it is necessary to define a new parameter more suitable for one static scale, multi-vernier scale residue verniers. This parameter is

$(N_{vs})_j$ = number of intervals on the i^{th} vernier scale per static scale cycle

where $i = 1, 2, \ldots, k$

This change in reference from the vernier scale cycle to the static scale cycle (N_{sv} to N_{sv}) also necessitates a change in the unit length. In the development for a two-scale device the unit length was found to be given by

$$L_x = \frac{L_S}{N_s N_v} \tag{51}$$

If the vernier and static scales were physically interchanged, then Eq (51) would become

$$L_x = \frac{L_V}{N_s N_v} \tag{108}$$

A moment's thought shows that for a two-scale device the change in the parameters from N_{sv} to N_{vs} is equivalent to the interchange of scales. Hence, Eq (108) gives the unit length for a two-scale device when the static scale is used as a reference.

When several vernier scales operate with one static scale and there is a common unit length, it follows from Eq (108) that

$$N_s L_x = L_{v_i} = L_v \tag{109}$$

Also, it follows from the definition of $(N_{vs})_i$ and Eq (109) that

$$(N_{sv})_i = \frac{N_s L_s}{L_{v_i}} = \frac{N_s L_s}{L_v} = N_{vs} \tag{110}$$

Thus, the parameters N_{vs} and L_v are common for all the vernier scales.

From Eq (53) of Chapter IV, it may be seen that the condition necessary for a one-static scale, multi-vernier scale residue vernier to satisfy the requirement $x = x'$ is

$$b_i L_{V_i} = aL_S \tag{111}$$

For multi-vernier scale devices the convenient assumption is that $a = 1$. Hence,

$$b_i L_{V_i} = L_S \tag{112}$$

Substitution of Eq (110) into Eq (112) gives

$$b_i N_{v_i} L_v = N_s \left(\frac{N_{vs} L_v}{N_s}\right) \tag{113}$$

which reduces to

$$b_i N_{v_i} = N_{vs} \tag{114}$$

Just as there was an implicit restriction on the factor a, so is there a restriction on the values b may have. This restriction on b may be developed from the condition required to prevent ambiguity of scale readings by logic very similar to that used for a. For devices using the static scale as a reference, the condition required to prevent ambiguity is

$$(N_s, N_{vs}) = 1 \tag{115}$$

The substitution of Eq(114) gives

$$(N_s, b_i N_{v_i}) = 1 \tag{116}$$

If $$(N_s, N_{v_i}) = 1 \tag{117}$$

and $$(N_s, b_i) = 1 \tag{118}$$

then from number theory (Chapter IV) Eq (116) is satisfied. Since the residue number system used with a vernier device requires the condition expressed by Eq (117), a sufficient condition for satisfying Eq (116) is that b_i be relatively prime to N_s. Hence, b_i must not only satisfy Eq (114), but must also satisfy Eq (118). A sufficient condition for meeting

these requirements on b_i is

$$b_i = \frac{C_s V}{N_{v_i}} \tag{119}$$

where

$$V \triangleq \prod_{i=1}^{k} N_{v_i} \tag{120}$$

and C_s is defined as a positive integer which satisfies

$$(C_s, N_s) = 1 \tag{121}$$

Substitution of Eq (119) into Eq (114) gives

$$N_{vs} = C_s V \tag{122}$$

From Eqs (108), (110), and (122) it may be seen that

$$L_s = \frac{N_{vs} L_v}{N_s} = C_s V L_x \tag{123}$$

<u>Coding Sequences</u>. The above expressions for the device parameters may now be applied to the basic two-scale device distance equations in order to develop the coding sequences. Hence,

$$xL_x = h_i L_s + m_i L_s - n_i L_v \tag{63}$$

Substitution of Eqs (109) and (123) gives

$$x = h_i C_s V + (m_i C_s V - n_i N_s) \qquad (124)$$

As before, the right-hand term provides the coding sequence. Thus, for the vernier scales

$$\left\{v_{n_i}\right\} = \left\{q(n_i)\ N_{v_i} - n_i N\right\} \qquad (125)$$

where: $n_i = 0, 1, 2, \ldots, N_{v_i}$ $\qquad 0 \leq v_{n_i} < N_{v_i}$

$$q(n_i) = \frac{V\ (m_i C_s - n_i)}{N_{v_i}} \qquad N = N_s - V$$

The other distance equation gives

$$x' L_x = dL_v + eL_v - fL_s \qquad (74)$$

The substitution of Eqs (77), (109), and (123) results in

$$x' = dN_s + \left[(e - C_s V + C_s N)\ N_s - nC_s\ (N_s - V)\right] \qquad (126)$$

where the prime has been dropped from n'. Hence, for the static scale the coding sequence is

$$\{s_n\} = \{q(n)\ N_s - nC_s N\} \tag{127}$$

where: $n = 0, 1, 2, \ldots, N_s$ $\quad 0 \leq s_n < N_s$

$q(n) = e - C_s V + C_s n$ $\quad N = N_s - V$

From prior logic it follows from Eqs (124) and (125) and Eqs (126) and (127) that at a coincidence of scales for a one static scale, multi-vernier scale residue vernier

$$x = |x|_{N_{v_i}} + N_{v_i} F_i(x) \tag{128}$$

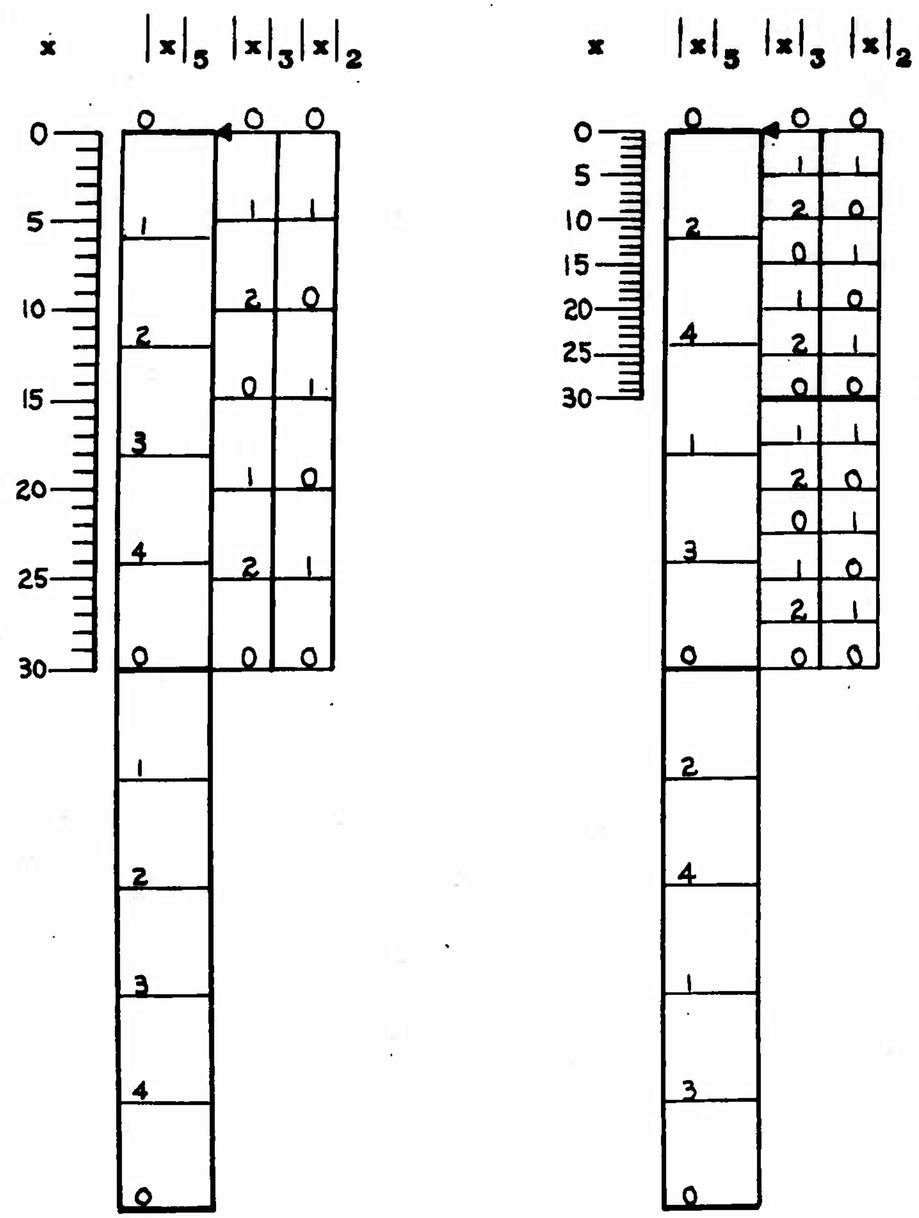
$$x = |x|_{N_s} + N_s\ G(x) \tag{129}$$

where $\quad F_i(x) = \dfrac{h_i C_s V}{N_{v_i}} \quad G(x) = d$

Thus, by the Chinese Remainder Theorem

$$x \longleftrightarrow (|x|_{N_s}, |x|_{N_{v_1}}, |x|_{N_{v_2}}, \ldots, |x|_{N_{v_k}}) \tag{130}$$

where $\quad 0 \leq x < N_s V$



(a) $C_s = 1$ $N_s = 5$

$N_{v_1} = 3$ $N_{v_2} = 2$

(b) $C_s = 2$ $N_s = 5$

$N_{v_1} = 3$ $N_{v_1} = 2$

Fig. 22

Multi-Vernier Scale

Residue Verniers

Figure 22 gives some simple examples of multi-vernier scale residue verniers. It should be noted that C_s also specifies the cyclic nature of the reference scale, but in a different way than does C_v. The parameter C_v determines the number of reference scale cycles per vernier scale cycle, whereas C_s determines the number of reference scale cycles per static scale cycle.

Multi-Scale Devices

Device Parameters. Sufficient insight has now been achieved to allow the generalization of the residue vernier relationships for a multi-vernier scale, multi-static scale device. A prerequisite for the design of the multi-scale device is a common L_x for all scales, a common L_v for the vernier scales, and a common L_s for the static scales. These common lengths are needed to insure the coincidence of all scales when the relative displacement is integral multiples of L_x. With this prerequisite it may be seen from Eqs (100) and (123) that

$$L_x = \frac{L_v}{C_v S} = \frac{L_s}{C_s V} \tag{131}$$

Hence,

$$L_v = C_v S L_x \tag{132}$$

$$L_s = C_s V L_x \tag{133}$$

However, before these expressions are complete it is necessary to determine if there has been any change in the constraints

placed on C_v and C_s as a result of attaching additional scales.

The determination of the constraints on C_v and C_s requires the use of two additional design prerequisites: the relation $x = x'$ must be satisfied and the device parameters must be of a value that prevents ambiguity of scale readings. From Eq (53) it may be seen that the necessary condition for meeting the relation $x = x'$ is

$$b_i L_{v_i} = a_j L_{s_j} \tag{134}$$

which may be rewritten as

$$b_i N_{v_i} L_v = a_j N_{s_j} L_s \tag{135}$$

In prior development it has been possible to reduce this expression by use of the parameters N_{sv} and N_{vs}. Examination of Figs. 21 and 22 reveals that for a multi-scale device these parameters are no longer integers; therefore, they are of little use in this generalized development. The length factors, however, may be removed from Eq (135) by the substitution of Eqs (132) and (133).

$$b_i N_{v_i} C_v S = a_j N_{s_j} C_s V \tag{136}$$

Figure 23 illustrates a vernier scale and a static scale of a multi-scale device. It may be seen from this figure that the condition necessary to prevent ambiguity of scale readings is

$$(a_j N_{s_j}, b_i N_{v_i}) = 1 \tag{137}$$

Since the Chinese Remainder Theorem requires that the moduli be relatively prime in pairs, it is necessary that

$$(N_{v_i}, N_{s_j}) = 1 \tag{138}$$

If

$$a_j = \frac{C_v S}{N_{s_j}} \tag{139}$$

where

$$(N_{v_i}, C_v) = 1 \tag{140}$$

then from Eqs (138) and (139)

$$(N_{v_i}, a_j N_{s_j}) = 1 \tag{141}$$

Also, if

$$b_i = \frac{C_s V}{N_{v_i}} \tag{142}$$

where

$$(C_s, a_j N_{s_j}) = 1 \tag{143}$$

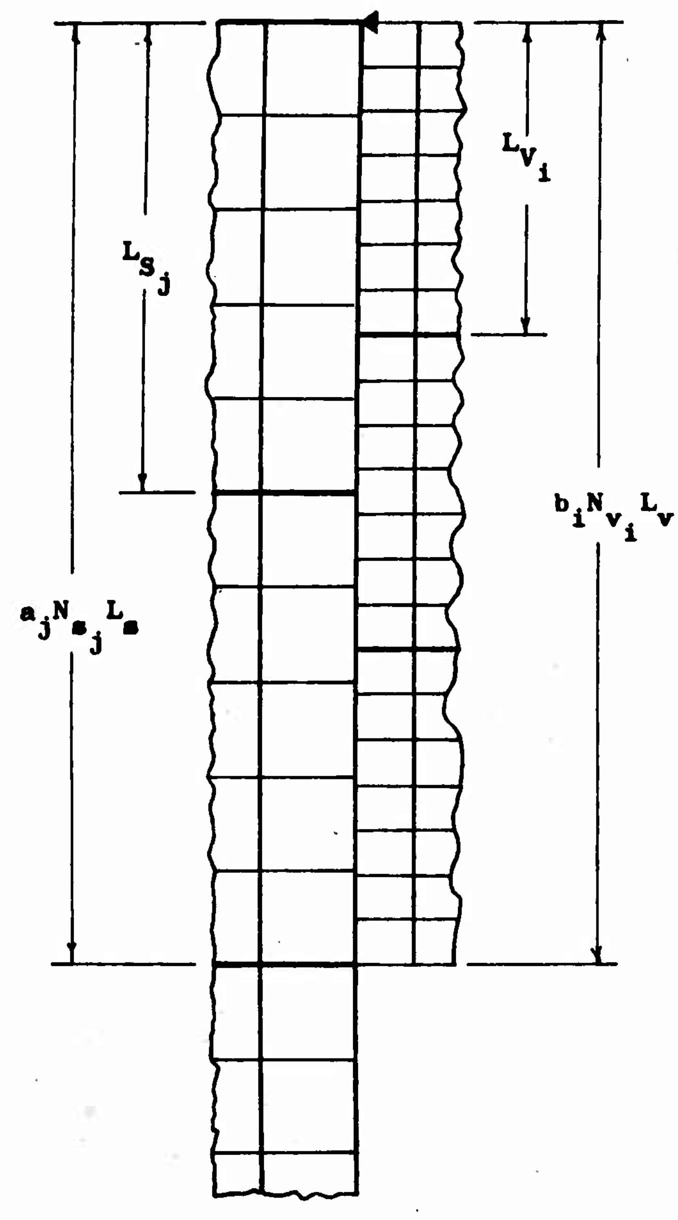


Fig. 23
Dimensions Of A Multi-Scale
Residue Vernier

then from Eqs (141) and (143)

$$(b_iN_{v_i}, a_jN_{s_j}) = 1 \tag{144}$$

Hence, the expressions given in Eqs (139) and (142) satisfy Eq (136), and Eqs (140) and (143) along with Eq (138) form a set of sufficient conditions for satisfying Eq (137).

There is another set of sufficient conditions that may be developed. If

$$b_i = \frac{C_sV}{N_{v_i}} \tag{142}$$

where $$(C_s, N_{s_j}) = 1 \tag{145}$$

then from Eqs (138) and (145)

$$(b_iN_{v_i}, N_{s_j}) = 1 \tag{146}$$

Also, if $$a_j = \frac{C_vS}{N_{s_j}} \tag{139}$$

where $$(b_iN_{v_i}, C_v) = 1 \tag{147}$$

then from Eqs (146) and (147)

$$(b_i N_{v_i}, a_j N_{s_j}) = 1 \tag{148}$$

Thus, Eqs (145) and (147) along with Eq (138) also form a set of sufficient conditions for satisfying Eq (137).

With the use of Eqs (139) and (142) the two sets of conditions imposed on C_v and C_s may be written as

$$(C_v, N_{v_i}) = 1 \qquad (C_s, C_v S) = 1 \tag{149}$$

or

$$(C_v, C_s V) = 1 \qquad (C_s, N_{s_j}) = 1 \tag{150}$$

Coding Sequences. As previously mentioned, it is necessary within the sets of static and vernier scales that the interval length be common. Because of this requirement, it is possible to consider a multi-static scale, multi-vernier scale device as being a combination of several two-scale residue verniers which have a common unit length. Thus, as in the previous development, it is permissible to use the distance equations developed for the two scale device. From Chapter IV

$$xL_x = h_i L_s + m_i L_s - n_i L_v \tag{63}$$

Substitution of Eqs (132) and (133) into Eq (63) gives

$$x = \left(\frac{h_i C_s V}{N_{v_i}}\right) N_{v_i} + \left[\frac{V(m_i C_s - n_i C_v)}{N_{v_i}}\right] N_{v_i} - n_i C_v \ (S-V) \qquad (151)$$

Thus,

$$\left\{v_{n_i}\right\} = \left\{q(n_i)\ N_{v_i} - n_i C_v N\right\} \qquad (152)$$

and

$$x = |x|_{N_{v_i}} + N_{v_i}\ F_i(x) \qquad (153)$$

where: $n_i = 0, 1, 2, \ldots, N_{v_i}$ $\qquad 0 \leq v_n < N_{v_i}$

$$F_i(x) = \frac{h_i C_s V}{N_{v_i}} \qquad N = S - V$$

$$q(n_i) = \frac{V(m_i C_s - n_i C_v)}{N_{v_i}}$$

From Eq (88) it may be seen that $\left\{v_{n_i}\right\}$ will be a monotone sequence when

$$C_v N = r_i N_{v_i} \mp 1 \qquad (154)$$

The second distance equation developed in Chapter IV may be written

$$x' L_x = d_j L_v + e_j L_v - f_j L_s \qquad (74)$$

Substitution of Eqs (77), (132), and (133) into Eq (74) gives

$$x' = \left(\frac{d_j C_v S}{N_{s_j}}\right) N_{s_j} + \left(\frac{e_j C_v S}{N_{s_j}} - C_s V + \frac{n_j C_s S}{N_{s_j}}\right) N_{s_j} - n_j C_s (S-V) \quad (155)$$

Thus,

$$\left\{s_{n_j}\right\} = \left\{q(n_j)\, N_{s_j} - n_j C_s N\right\} \quad (156)$$

and

$$x = |x|_{N_{s_j}} + N_{s_j}\, G_j(x) \quad (157)$$

where: $n_j = 0,1,2, \ldots, N_{s_j}$ $\qquad 0 \leq s_{n_j} < N_{s_j}$

$$G_j(x) = \frac{d_j C_v S}{N_{s_j}} \qquad N = S - V$$

$$q(n_j) = \frac{e_j C_v S}{N_{s_j}} - C_s V + \frac{n_j C_s S}{N_{s_j}}$$

The requirement for monotonicity of $\left\{s_{n_j}\right\}$ may be developed by applying to Eq (156) the same logic used in Chapter II, Eqs (19) through (22). The resulting requirement is

$$C_s N = r_j N_{s_j} \mp 1 \quad (158)$$

The Chinese Remainder Theorem applied to the sets of equations given by Eq (153) and (157) gives

$$x \leftrightarrow \left(|x|_{N_{v_1}}, |x|_{N_{v_2}}, \ldots, |x|_{N_{v_k}}, |x|_{N_{s_1}}, |x|_{N_{s_2}}, \ldots, |x|_{N_{s_m}}\right) \quad (159)$$

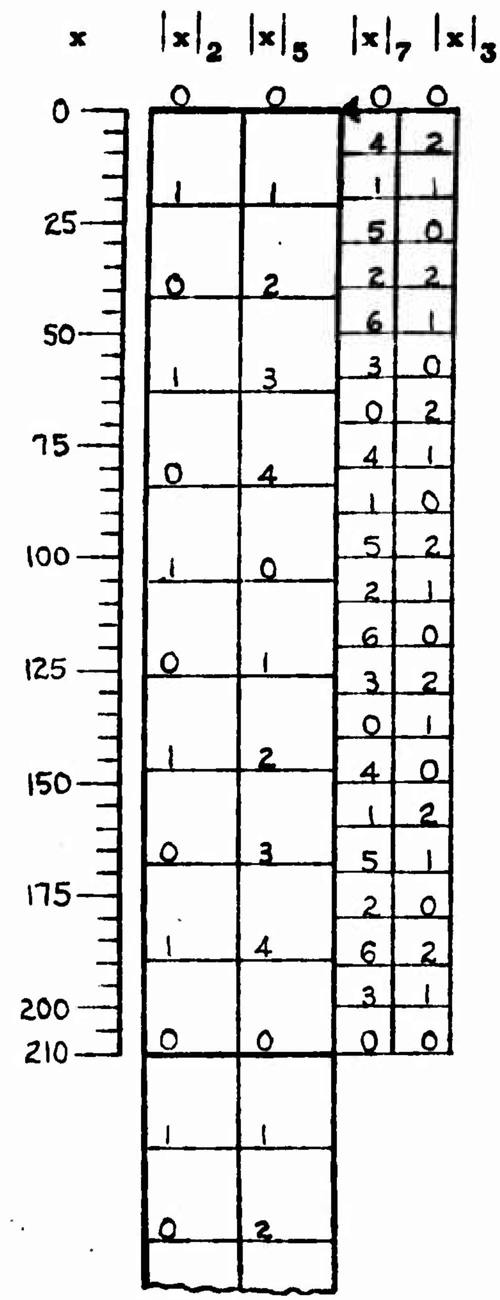


Fig. 24

Multi-Scale

Residue Vernier

where $$0 \leq x < VS$$

The preceding development has proven that a multi-vernier scale, multi-static scale device, when properly designed, will at coincidence of scales provide the residue number representation of x. A simple example (in fact, it's the simplest case) of such a vernier device is given in Fig. 24.

There is one obvious point that is sufficiently important to rate a mention. This point is that the above generalized expressions reduce to the expressions developed earlier for specific cases.

VI. Summary of Pertinent Results

The purpose of this paper, as stated in Chapter I, was to investigate the relationship between vernier devices and residue number systems. This investigation is now complete.

In order to conduct the investigation, it was found necessary to first develop a theory for conventional verniers. The results of this initial work are summarized at the end of Chapter II. With a theory of conventional verniers developed, it was then possible to investigate the relationship between vernier devices and residue numbers. The principal results of this investigation are the design requirements for the residue vernier, a device for converting numbers in decimal form into their residue equivalent and vice versa. These results are summarized below:

1. The moduli of the residue number system to be used determine the device parameters N_{v_i} and N_{s_j}. To prevent ambiguity in the number system, it is necessary that

$$(N_{v_i}, N_{s_j}) = 1 \tag{138}$$

where $i = 1,2, \ldots, k$ and $j = 1,2, \ldots, m$

2. Given the above relation, there are two sets of sufficient conditions for preventing ambiguity of scale readings.

These are

$$(C_v, N_{v_i}) = 1 \qquad (C_s, C_v S) = 1 \tag{149}$$

or

$$(C_v, C_s V) = 1 \qquad (C_s, N_{s_j}) = 1 \tag{150}$$

where

$$S = \prod_{j=1}^{m} N_{s_j} \qquad V = \prod_{i=1}^{k} N_{v_i} \qquad (97)\ (120)$$

and C_v and C_s determine the cyclic nature of the reference scale.

3. The length of the intervals on the vernier and static scales are given by

$$L_v = C_v S L_x \qquad L_s = C_s V L_x \qquad (132)\ (133)$$

4. The coding sequences for the vernier scales are

$$\left\{v_{n_i}\right\} = \left\{q(n_i)\, N_{v_i} - n_i C_v N\right\} \tag{152}$$

where $\quad n_i = 0,1,2,\ \ldots,\ N_{v_i} \qquad 0 \leq v_{n_i} < N_{v_i}$

and

$$N = S - V$$

5. The coding sequences for the static scales are

$$\left\{s_{n_j}\right\} = \left\{q(n_j)\, N_{s_j} - n_j C_s N\right\} \tag{155}$$

where $\quad n_j = 0,1,2, \ldots, N_{s_j} \qquad 0 \leq s_{n_j} < N_{s_j}$

6. Given parameters N_{v_i}, N_{s_j}, C_v, and C_s which meet the conditions given in paragraphs 1 and 2, then a vernier device constructed and coded according to the requirements given in paragraphs 3, 4, and 5 will have scale readings at coincidence that

$$x \longleftrightarrow \left(|x|_{N_{v_1}}, |x|_{N_{v_2}}, \ldots, |x|_{N_{v_k}}, |x|_{N_{s_1}}, |x|_{N_{s_2}}, \ldots, |x|_{N_{s_m}} \right) \quad (159)$$

where $\qquad 0 \leq x < VS$

7. If the device parameters are such that

$$C_v N = r_i N_{v_i} \mp 1 \quad (154)$$

then the code sequence for the particular vernier scale will be monotone. For static scales if

$$C_s N = r_j N_{s_j} \mp 1 \quad (158)$$

then the particular static scale will have a monotone code sequence.

8. For linear verniers, the minimum ratio of static scale length to vernier scale length required to provide the complete range of x is

$$1 < \frac{N_S L_S}{N_V L_V} < 2 \tag{89}$$

VII. Recommendations for Further Work

During the course of investigating the relationship between vernier devices and residue numbers, several areas requiring additional work became apparent. Unfortunately, due to a time limitation the author was unable to pursue these further. The following list presents some of these areas:

1. An immediately apparent application for the theory developed in this paper is for input-output translations. It appears that the vernier approach is applicable to both decimal and analog conversion. Techniques for performing these conversions using this approach should be investigated.

2. It has been shown that for certain operations with residue numbers it is desirable to have a knowledge of the function $W(x)$ which was introduced in Chapter III (Ref 1: Chap III, p. 8-16 and 3:86-90). It appears that the relative displacement of the vernier scale is associated with this function. This relationship should be investigated and developed.

3. The theory developed in this paper has been necessarily tied to a mechanical device which has inherent disadvantages. An investigation could be performed to determine the possibility of realizing the vernier function electronically. One approach that may be possible would be to use coincidence of pulses; the pulse frequency corresponding to the moduli of number system.

Some type of time delay may possibly be used to simulate relative scale displacement.

4. A direct extension of the work done in this paper would be an investigation of the application of residue verniers to residue number arithmetic.

5. It was noted in Chapter II that vernier devices could be constructed so that the scale readings give x in either a fixed or a mixed radix system. This requires investigation to determine what, if any, constraints exist and what applications are possible.

6. As indicated in Chapter VI, some of the conditions for designing a residue vernier are only sufficient. Further work could be done to determine the necessary and sufficient conditions.

Bibliography

1. Aiken, H. and W. Semon. Advanced Digital Computer Logic. WADC TR 59-472. Wright-Patterson AFB, Ohio: Wright Air Development Center, USAF. July 1959.

2. Garner, H.L., et al. Residue Number Systems for Computers. ASD TR 61-483. Wright-Patterson AFB, Ohio: Aeronautical Systems Division, USAF, October, 1961.

3. Lockheed Missiles and Space Co. Interim Technical Report on Modular Arithmetic Techniques. ASD TR 61-472. Wright-Patterson AFB, Ohio: Aeronautical Systems Division, USAF, December 1961.

4. Ore, Oystein. Number Theory and Its History. New York: McGraw-Hill Book Co., Inc., 1948.

Vita

Richard Earl Evans was born on 20 January 1929 in Shawnee, Oklahoma, the son of Laurence Richard Evans and Mary Shelton Evans. After completing his work in 1946 at Capitol Hill High School, Oklahoma City, Oklahoma, he enlisted in the U.S. Navy. Upon his discharge in 1948, he enrolled in the University of Oklahoma. With the advent of the Korean conflict, he enlisted in the U.S. Air Force in 1950. He served as an enlisted man until 1953, at which time he graduated from Officer Candidate School and was commissioned as a Lieutenant in the USAF. His military assignments prior to his coming to the Air Force Institute of Technology included the Air Training Command and the Headquarters Command and were primarily in the field of electronics.

Permanent address: 608 South Court
Visalia, California

This thesis was typed by Mrs. Marie A. Evans.